ENUMERATION OF PARTITIONS SUBJECT TO LIMITATIONS ON SIZE OF MEMBERS

Daniel C. Fielder, Georgia Institute of Technology, Atlanta, Georgia

1. INTRODUCTION

In a previous work [1], it was shown that the partition enumeration* $P(n|p| \le n + p - 1)$ is given by

$$(1) \qquad P(n|p| \le n + p - 1) = \left[\frac{n - p + 2}{2}\right] + \sum_{i=1} \left[\frac{n - p + 2 - w_i}{2}\right] \qquad p \neq 1,$$

$$(1a) \qquad P(n|p| \le n + p - 1) = 1 \qquad p = 1.$$

The w_i are the sums of each partition in the set of partitions described by $PV(\ge 3, \le n - p| \ge 1, \le [(n - p)/3]| \ge 3, \le p)$. It was stated in [1] that the summation term of (1) is zero for those values of p and (n - p) for which $PV(\ge 3, \le n - p| \ge 1, \le [(n - p)/3]| \ge 3, \le p)$ does not exist. (See the footnote below for a brief description of nomenclature.) For $n - p < 3$ and/or $p = 2$, $w_i = 0$. One <u>raison d'être</u> for (1) is the adaptability of w_i to digital computation.

$P(n|p| \le n - p + 1)$ is a basic enumeration form which is extremely useful in evaluating more restrictive enumerations [2]. $PV(n|p| \le n - p + 1)$ shares this versatility in that sets of many other partition types can be constructed by operations on the members of the partitions of the basic set. When $PV(n|p| \le n - p + 1)$ is under consideration, it is convenient to arrange the p members of a partition so that

$$(2) \qquad a_p \le a_{p-1} \le a_{p-2} \le \cdots \le a_2 \le a_1 \quad ,$$

$P(n|p| \le q)$ is the enumeration of the partitions of n into exactly p members, no member of which is greater than q. The appended notation $PV(n|p| \le q)$ is the actual set of such partitions. The use of $\ge$ and/or $\le$ symbols with n, p, or q defines lower limits and/or upper limits of the quantity modified. Note that [ ] (except for obvious reference use) is used with real numbers to indicate the greatest integer less than or equal to the number bracketed.

where a_k is an individual partition member. The arrangement of (2) leads to an initial partition of $PV(n|p| \le n - p + 1)$ as

(3)
$$\begin{array}{c|c|c|c|c|c} a_p & a_{p-1} & a_{p-2} & \cdots & a_2 & a_1 \\ \hline 1 & 1 & 1 & \cdots & 1 & n - p + 1 \end{array} \quad .$$

One method [3] of generating successive partitions of $PV(n|p| \le n - p + 1)$ starts with (3) and successively increases a_2 by 1 and decreases a_1 by 1 until (2) is just barely satisfied. New members, a_p, $a_{p-1}, \cdots, a_2, a_1$ are chosen exhaustively, and the increase a_2—decrease a_1 process is repeated.

Based on the above brief background, it is possible to consider the following enumeration extensions to (1):

(a) $P(n|p| \ge s)$. No member less than s, where s is a positive integer such that $s \le n - p + 1$.

(b) $P(n|p| \le r)$. No member greater than r where r is a positive integer such that $r \le n - p + 1$.

(c) $P(n|p| \ge s, \le r)$. No member less than s, or greater than r .

2. ENUMERATION OF $P(n|p| \ge s)$

There exists one member of a partition in the set $PV(n|p| \ge s)$ which is at least as large as any member of any partition in the set. Let this member be q_s which can readily be found as

(4)
$$q_s = n - s(p - 1) \ .$$

This implies that for any a_1,

(5)
$$n - ps + s \ge a_1 \ge s \quad ,$$

from which a necessary condition of $P(n|p| \ge s)$ is seen to be

(6)
$$\left(\frac{n}{p}\right) \ge s \ .$$

The initial partition of $PV(n|p|\geq s)$ is

$$s, s, s, \cdots, s, n - s(p - 1) \tag{7}$$

If $s - 1$ is subtracted from each member of (7), the result is a modified initial partition

$$1, 1, 1, \cdots, 1, n - sp + 1 \; . \tag{8}$$

The complete enumeration for a partition set starting with (8) is, according to (1), $P(n'|p|\leq n' - p + 1)$, where

$$n' = n - sp + p \; . \tag{9}$$

Because the a_1 and a_2 members of the initial partitions (7) and (8) differ by the same integer $(n - sp)$ and because each a_k of each partition developed from (7) is $(s - 1)$ greater than the corresponding a_k of the corresponding partition developed from (8), there are exactly as many partitions developable from the start of (7) as there are from (8). Hence, $P(n|p|\geq s)$ appears in the form of (1) as

$$P(n|p|\geq s) = P(n'|p|\leq n' - p + 1) \; . \tag{10}$$

As a simple example, consider $P(15|6|\geq 2)$. For this case, $n = 15$, and $n' = 9$. It is seen below that $P(15|6|\geq 2) = P(9|6|\leq 4) = 3$.

$PV(15\|6\|\geq 2)$	$PV(9\|6\|\leq 4)$
2, 2, 2, 2, 2, 5	1, 1, 1, 1, 1, 4
2, 2, 2, 2, 3, 4	1, 1, 1, 1, 2, 3
2, 2, 2, 3, 3, 3	1, 1, 1, 2, 2, 2

3. ENUMERATION OF $P(n|p|\leq r)$

The partitions of the set $PV(\geq 3, \leq n - p|\geq 1, \leq [(n - p)/3]\,|\geq 3 \leq p)$ can be arranged in columns according to the number of members in a partition. This is illustrated in Table 1 for $n = 16$, $p = 5$.

i	0	1	2	3
$PV(\geq 3, \leq 11 \mid \geq 1, \leq 3 \mid \geq 3, \leq 5)$	0	3	3,3	3,3,3
		4	3,4	3,3,4
		5	3,5	3,3,5
			4,4	3,4,4
			4,5	
			5,5	

Table 1

The sum of members of each partition is equal to a w_i for use in (1). The use of the index i can be extended somewhat to allow it to designate the column from which the summed partition was taken. Although w_i might stand for any of several sums, no loss in generality results thereby since all of these sums must eventually be considered. To account for the non-summation term in (1), a zeroth column with a lone zero entry is added to indicate that an added $w_0 = 0$. Table 2 shows values of w_i for $n = 16$, $p = 5$.

i	0	1	2	3
	0	3	6	9
		4	7	10
		5	8	11
			8	11
			9	
			10	

Table 2 Values of w_i

If, as the w_i's are successively selected for enumerating $P(n|p| \leq n - p + 1)$ in (1), a simultaneous generation of the partitions in the set $PV(n|p| \leq n - p + 1)$ is made (by the increase a_2—decrease a_1 method, for example) there would result subsets of $PV(n|p| \leq n - p + 1)$ each having $[(n - p + 2 - w_i)/2]$ partitions of n. For $i = 0$, the subset can easily be constructed. It is seen that the a_2 and a_3 members of the initial partition must necessarily be one. For $i = 1$, the a_2 and a_3 members of the initial partition assume the least possible value two since $i = 0$ has accounted for the value one. It can be argued in this fashion that the a_2 and a_3 members of an initial partition in a subset must be $(i + 1)$. The a_1 member of the initial partition of the subset would not generally be known in advance. However, this member is certainly not less than any member of any partition in the subset. Set d_i be the a_1 member of the initial partition corresponding to the particular w_i. If b_i are the number of partitions in the subset, the bracketed terms of (1) limit the possibilities of b_i to either

(11) $$n - p + 1 - w_i = 2b_i - 1 \quad ,$$

or

(11a) $$n - p + 1 - w_i = 2b_i \quad .$$

The arrangement of the subset of b_i partitions is

(12) $$\begin{array}{cc} & \begin{array}{cccccc} a_p & a_{p-1} & \cdots & a_3 & a_2 & a_1 \\ \hline \end{array} \\ \begin{array}{c} b_i \\ \text{partitions} \end{array} & \left\{ \begin{array}{cccccc} x & x & \cdots & i+1 & i+1 & d_i \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ x & x & \cdots & i+1 & i+b_i & d_i-b_i+1 \end{array} \right. \end{array} \begin{array}{l} \leftarrow \text{(Initial Partition)} \\ \\ \\ \\ \\ \end{array}$$

From (12), it can be deduced that either

(13) $$d_i = 2b_i - 1 + i \quad ,$$

or

(13a) $$d_i = 2b_i + i \quad .$$

Comparison of (13) with (11) and (13a) with (11a) yields the desired

(14) $$d_i = (n - p + 1 - w_i + i) \quad .$$

An illustration is given in Table 3 for construction of PV(16|5|≤12), consistent with the w_i from PV(≥3,≤11|≥1,≤3|≥3,≤5) as arranged in Tables 1 and 2.

i	0	1	2	3
	1,1,1,1,1,12	1,1,2,2,10	1,1,3,3,8	1,1,4,4,6
	1,1,1,1,2,11	1,1,2,3, 9	1,1,3,4,7	1,1,4,4,5
	1,1,1,1,3,10	1,1,2,4, 8	1,1,3,5,6	
	1,1,1,1,4, 9	1,1,2,5, 7		
	1,1,1,1,5, 8	1,1,2,6, 6		
	1,1,1,1,6, 7			
		1,2,2,2, 9	1,2,3,3,7	1,2,4,4,5
		1,2,2,3, 8	1,2,3,4,6	
		1,2,2,4, 7	1,2,3,5,5	
PV(16|5|≤12)		1,2,2,5, 6		
		2,2,2,2, 8	1,3,3,3,6	1,3,4,4,4
		2,2,2,3, 7	1,3,3,4,5	
		2,2,2,4, 6		
		2,2,2,5, 5		
			2,2,3,3,6	2,2,4,4,4
			2,2,3,4,5	
			2,3,3,3,5	
			2,3,3,4,4	
			3,3,3,3,4	

Table 3 PV(16|5|≤12)

Table 4 shows b_i corresponding to w_i of Table 1 for $P(16|5|\le 12) = \sum_{i=0} b_i$.

i	0	1	2	3
	6	5	3	2
		4	3	1
		4	2	1
b_i			2	1
			2	
			1	

Table 4 $\sum_i b_i = P(16|5|\le 12) = 37$

Let the b_i for $P(n|p|\leq r)$ be b_{ir}. It follows that for $P(n|p|\leq r)$ each b_{ir} can have no more (and will possibly have less) than b_i partitions. The non-negative integer by which b_{ir} is less than b_r can be observed by comparing r with the entries in the a_1 column of (12). This leads immediately to

$$(15)\quad P(n|p|\leq r) = \left[\frac{n - p + 2}{2}\right] - \alpha_0 + \sum_{i=1}\left(\left[\frac{n - p + 2 - w_i}{2}\right] - \alpha_1\right) = \sum_{i=0} b_{ir} ,$$

where

$$(16)\quad \alpha_1 = \begin{cases} 0 & (r \geq d_i) , \\ d_i - r & (d_i > r \geq (d_i - b_i + 1)) , \\ \left[\dfrac{n - p + 2 - w_i}{2}\right] & (r < (d_i - b_i + 1)). \end{cases}$$

Table 5 serves to illustrate (15) for n = 16, p = 5, r = 7.

i	0	1	2	3
b_{ir}	1	2	2	2
		2	3	1
		3	2	1
			2	1
			1	

Table 5 $\sum_i b_{ir} = P(16|5|\leq 7) = 23$

4. ENUMERATION OF $P(n|p|\geq s, \leq r)$

The combination of the previous two methods leads quickly to the desired enumeration. Reference to (10) reveals a $P(n'|p|\leq n' - p + 1)$ for which every member of each partition of $PV(n'|p|\leq n' - p + 1)$ is (s - 1) less than the corresponding member of the appropriate counterpart in $PV(n|p|\geq s)$. If the desired r is depressed to r' where

(17) $$r' = r - (s - 1) \quad ,$$

the enumeration $P(n, |p| \geq s, \leq r)$ is equal to $P(n'|p| \leq r')$.

REFERENCES

1. D. C. Fielder, "Partition Enumeration by Means of Simpler Partitions," Fibonacci Quarterly, Vol. 2, No. 2, pp 115—118, 1964.
2. G. Chrystal, Textbook of Algebra, Vol. 2 (Reprint) Chelsea Publishing Co., New York, N.Y., pp 555—565, 1952.
3. D. C. Fielder, "A Combinatorial-Digital Computation of a Network Parameter," IRE Trans. on Circuit Theory, Vol. PGCT-8, No. 3, pp 202—209, Sept. 1961.

* * * * *

DID YOU KNOW?

Prof. D. E. Knuth of California Institute of Technology is working on a 3-volume book, The Analysis of Algorithms, which has 39 exercises at the end of the section which introduces the Fibonacci Sequence. However, the Fibonacci Sequence occurs in many different places, both as an operational tool, or to serve as examples of good sequences and also bad sequences. He reports that there are at least 12 different algorithms directly or indirectly connected with the Fibonacci Sequence. In the age of computers, the Fibonacci Sequence is coming of age in many ways. This book will be a most welcome addition to the growing list of Fibonacci related books and articles.

*

Prof. C. T. Long of Washington State University has written a very nice book, Elementary Introduction to Number Theory, 1965, Heath, Boston. It contains a good discussion of the Fibonacci Numbers in Chapter One and several Fibonacci Problems in Chapters I and II.

* * * * *