A SINGULAR FIBONACCI MATRIX AND ITS RELATED LAMBDA FUNCTION

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After a very brief introduction to some of the extremely basic properties of Fibonacci numbers, a student of mine inductively produced the following identities concerning determinants of Fibonacci matrices:

$$(1) \qquad \begin{vmatrix} F_n & F_{n+1} \\ F_{n+s+2} & F_{n+s+3} \end{vmatrix} = (-1)^{n+1} F_{s+2}$$

$$(2) \qquad \begin{vmatrix} F_n & F_{n+m+1} \\ F_{n+m+2} & F_{n+2m+3} \end{vmatrix} = (-1)^{n+1} F_{m+1} F_{m+2}$$

$$(3) \qquad \begin{vmatrix} F_n & F_{n+m+1} \\ F_{n+m+s+2} & F_{n+2m+s+3} \end{vmatrix} = (-1)^{n+1} F_{m+1} F_{m+s+2} \quad .$$

Each row of the determinant is regarded as a pair of numbers, the subscript s refers to the number of terms in the Fibonacci sequence skipped between successive pairs, and the subscript m refers to the number of terms skipped between the two numbers of a pair.

It is simple exercise to establish the validity of (1), (2), and (3) using $F_{m+n} = F_{n-1}F_m + F_nF_{m+1}$. However, close inspection will show that (1), (2), and (3) are only special cases and/or variations of

$$(4) \qquad F_p F_q - F_{p-k} F_{q+k} = (-1)^{p-k} F_k F_{q+k-p} ,$$

where $k = m + 1$ and $q - p = s + 1$.

This comparison is made easier when (4) is written as

$$(4') \qquad \begin{vmatrix} F_{p-k} & F_p \\ F_q & F_{q+k} \end{vmatrix} = (-1)^{p-k+1} F_k F_{q+k-p}$$

thus suggesting a form for a related 3×3 matrix

$$P = \begin{bmatrix} F_{j-k} & F_j & F_{j+k} \\ F_{m-k} & F_m & F_{m+k} \\ F_{n-k} & F_n & F_{n+k} \end{bmatrix}$$

A singular property of the Det(P) presents itself.

Theorem: Det(P) = 0, k, j, m, n are integers.

Proof: There is no loss in generality to assume $j > m > n$ and it is simply convenient to assume $k \geq 0$. By applying (4) it is apparent that the columns of P are linearly dependent. We note by inspection that F_k (column 3) - F_{2k} (column 2) = $(-1)^{k+1} F_k$ (column 1). Thus, the determinant is clearly zero (0).

Q. E. D.

Since the Det(P) = 0, a previous article of this Quarterly [3] suggests it would be interesting to consider the Det(P + a) where P + a means a matrix P with a added to each element of P. The generality of j, m, n, and k would almost prohibit the techniques used by Whitney [3]. Hence procedures discussed by Bicknell in [1] and by Bicknell and Hoggatt in a previous article of this Quarterly [2] are employed. Using the formula [2]

$$\mathrm{Det}(P + a) = \mathrm{Det}(P) + a\,\lambda(P)\ ,$$

where $\lambda(P)$ is the change in the value of the determinant of P, when the number 1 is added to each element of P, we have

$$\mathrm{Det}(P + a) = a\,\lambda(P)\ ,$$

since Det(P) = 0. Now $\lambda(P)$ and the corresponding Det(P + a) are interesting in any one of the following forms. They are also derived with the aid of (4').

(a) $$\lambda(P) = \begin{vmatrix} 1 & 1 & 1 \\ F_{m-k} - F_{j-k} & F_m - F_j & F_{m+k} - F_{j+k} \\ F_{n-k} - F_{j-k} & F_n - F_j & F_{n+k} - F_{j+k} \end{vmatrix}$$

or

(b) $$\lambda(P) = \left[F_{2k} - F_k - (-1)kF_k\right]\left[(-1)^{m-k}F_{n-m} + (-1)^{j-k}F_{n-j} - (-1)^{j-k}F_{m-j}\right]$$

Therefore,

(c) $$\operatorname{Det}(P + a) = \begin{vmatrix} a & a & a \\ F_{m-k} - F_{j-k} & F_m - F_j & F_{m+k} - F_{j+k} \\ F_{n-k} - F_{j-k} & F_n - F_j & F_{n+k} - F_{j+k} \end{vmatrix}$$

or

(d) $$\operatorname{Det}(P + a) = [F_{2k} - F_k - (-1)^k F_k][(-1)^{m-k}F_{n-m} + (-1)^{j-k}F_{n-j} - (-1)^{j-k}F_{m-j}]\, a.$$

The first factors of (b) and (d) have a straightforward simplification if it is known in advance whether or not k is even or odd. The various forms of $\lambda(P)$ and $\operatorname{Det}(P + a)$ become much more intriguing once the interesting patterns in the subscripts and exponents and their relationship to P are observed. These patterns could easily serve as mnemonic devices.

REFERENCES

1. Marjorie Bicknell, "The Lambda Number of a Matrix: The Sum of Its n^2 Cofactors," Amer. Math. Monthly, 72 (1965), pp 260—264.
2. Marjorie Bicknell and V. E. Hoggatt, Jr., "Fibonacci Matrices and Lambda Functions," Fibonacci Quarterly, Vol. 1, No. 2, April 1964, pp 47—50.
3. Problem B-24, Fibonacci Quarterly, Vol. 2, No. 2, April, 1964.

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