ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning Advanced Problems and Solutions to Raymond Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within three months after publication of the problems.

H-93 ***Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*** (Corrected)

that

$$F_n = \prod_{k=1}^{\overline{n-1}} \left(3 + 2\cos\frac{2k\pi}{n}\right)$$

$$L_n = \prod_{k=1}^{\overline{n-2}} \left(3 + 2\cos\frac{(2k+1)\pi}{n}\right) \qquad ,$$

where $\bar{n}$ is the greatest integer in $n/2$.

H-96 ***Proposed by Maxey Brooke, Sweeny, Texas, and V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.***

Suppose a female rabbit produces F_n (L_n) female rabbits at the n^{th} time point and her female offspring follow the same birth sequence, then show that the new arrivals, C_n, (, D_n,) at the n^{th} time point satisfies

$$C_{n+2} = 2C_{n+1} + C_n \qquad C_1 = 1 \qquad C_2 = 2$$

and

$$D_{n+1} = 3D_n + (-1)^n \qquad D_1 = 1 \quad .$$

H-97 ***Proposed by L. Carlitz, Duke University, Durham, N.C.***

Show

(a)
$$\sum_{k=0}^{n} \binom{n}{k}^2 L_k = \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k}\binom{n+k}{k} L_{n-k}$$

(b)
$$\sum_{k=0}^{n} \binom{n}{k}^2 F_k = \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k}\binom{n+k}{k} F_{n-k} \quad .$$

H-98 **Proposed by George Ledin, Jr., San Francisco, Calif.**

If the sequence of integers is designated as J, the ring identity as I, and the quasi-inverse of J as F, then $(I - J)(I - F) = I$ should be satisfied. For further information see R. G. Buschman, "Quasi Inverses of Sequences," American Mathematical Monthly, Vol. 73, No. 4, III (1966) p. 134.

Find the quasi-inverse sequence of the integers (negative, positive, and zero).

H-99 **Proposed by Charles R. Wall, Harker Heights, Texas.**

Using the notation of H-63 (April 1965 FQJ, p. 116), show that if $\alpha = (1 + \sqrt{5})/2$,

$$\prod_{n=1}^{m} \sqrt{5} F_n \alpha^{-n} = 1 + \sum_{n=1}^{m} (-1)^{n(n-1)/2} F(n,m) \alpha^{-n(m+1)}$$

$$\prod_{n=1}^{m} L_n \alpha^{-n} = 1 + \sum_{n=1}^{m} (-1)^{n(n+1)/2} F(n,m) \alpha^{-n(m+1)} ,$$

where

$$F(n,m) = \frac{F_m F_{m-1} \cdots F_{m-n+1}}{F_1 F_2 \cdots F_n} .$$

H-100 **Proposed by D.W. Robinson, Brigham Young Univ., Provo, Utah.**

Let N be an integer such that $F_n \le N < F_{n+1}$, $n \ge 1$. Find the maximum number of Fibonacci numbers required to represent N as an Algebraic Sum of these numbers.

H-101 **Proposed by Harlan Umansky, Cliffside Park, N. J., and Malcolm Tallman, Brooklyn, N.Y.**

Let a, b, c, d be any four consecutive generalized Fibonacci numbers (say $H_1 = p$ and $H_2 = q$ and $H_{n+2} = H_{n+1} + H_n$, $n \ge 1$), then show

$$(cd - ab)^2 = (ad)^2 + (2bc)^2$$

Let $A = L_k L_{k+3}$, $B = 2L_{k+1} L_{k+2}$, and $C = L_{2k+2} + L_{2k+4}$. Then show

$$A^2 + B^2 = C^2 .$$

H-102 **Proposed by J. Arkin, Suffern, N. Y.**

Find a closed expression for A_m in the following recurrence relation.

$$\left[\frac{m}{2}\right] + 1 = A_m - A_{m-3} - A_{m-4} - A_{m-5} + A_{m-7} + A_{m-8} + A_{m-9} - A_{m-12},$$

where $m = 0, 1, 2, \cdots$ and the first thirteen values of A_0 through A_{12} are

1,1,2,3,5,7,10,13,18,23,30,37, and 47, and $[x]$ is the greatest integer contained in x.

SOLUTIONS

EULER AND FIBONACCI

H-54 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

If F_n is the n^{th} Fibonacci number, then show that

$$\phi(F_n) \equiv 0 \pmod 4, \qquad n > 4$$

where $\phi(n)$ is Euler's function.

Solution by John L. Brown Jr., Penn. State Univ., State College, Pa.

It is well known that $m|n$ implies $F_m|F_n$. Further, $F_n = 3 \pmod 4$ implies $n = 6k - 2$ for $k = 1,2,3,\cdots$. Therefore, if F_n is a prime and $n > 4$, F_n must be of the form $4s + 1$ with s a positive integer. [Otherwise, $F_n, n > 4$) would be of the form $4r + 3$ and hence $n = 6k - 2$ with $k \geq 2$ implying that $F_{3k-1}|F_n$, contrary to assumption.] Since $\phi(p) = p - 1$ for any prime p, it is therefore clear that F_n prime with $n > 4$ implies $\phi(F_n) \equiv \phi(4s + 1) \equiv (4s + 1) - 1 \equiv 4s \equiv 0 \pmod 4$.

Now, for any integer n,

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right),$$

where $p_1, p_2, \cdots, p_k$ are the distinct prime divisors of n. Therefore, for $n = ab$ with a and b integers,

$$\phi(ab) = ab\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right),$$

where $p_1, p_2, \cdots, p_k$ are the distinct prime divisors of ab. Since the distinct prime divisors of either a or b separately are included among those of ab, it is obvious that either $\phi(a) \equiv 0 \pmod 4$ or $\phi(b) \equiv 0 \pmod 4$ necessarily implies $\phi(ab) \equiv 0 \pmod 4$.

We shall now prove by an induction on n that $\phi(F_n) \equiv 0 \pmod 4$ for $n > 4$. First, the result is easily verified for $n = 5, 6, \cdots, 10$. Assume as an induction hypothesis that it has been proved for all $n \leq t$ where t is an integer ≥ 10. Then, if F_{t+1} is prime, we have $\phi(F_{t+1}) \equiv 0 \pmod 4$ by the result of the first paragraph. Otherwise, we distinguish 2 cases. If $t + 1$ is composite,

$t+1$ may be written as $t+1 = m_1 m_2$ where m_1 and m_2 are integers both >1, and at least one of them, say m_1 for definiteness, is >5 and $<t$. Then $F_{m_1} \mid F_{t+1}$, so that $F_{t+1} = F_{m_1} \cdot q$. Since $\phi(F_{m_1}) \equiv 0 \pmod 4$ by the induction hypothesis, we conclude from the remarks of the second paragraph that $\phi(F_{t+1}) \equiv \phi(F_{m_1} \cdot q) \equiv 0 \pmod 4$ as required.

In the alternative case where $t+1$ is prime, we note that F_{t+1} is odd (otherwise $t+1$ would be divisible by 3) and composite. Hence F_{t+1} has only odd prime factors, say $p_1, p_2, \cdots, p_k$, and

$$\phi(F_{t+1}) = F_{t+1}\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right).$$

Since $k \ge 2$, it is clear that

$$\phi(F_{t+1}) = \frac{F_{t+1}}{p_1 p_2 \cdots p_k}(p_1 - 1)(p_2 - 1)\cdots(p_k - 1)$$

is divisible by 4. Thus in all cases, $\phi(F_{t+1}) \equiv 0 \pmod 4$ and the proof is completed by mathematical induction.

Also solved by the proposer.

H-56 ***Proposed by L. Carlitz, Duke University, Durham, N.C.***

Show

$$\sum_{n=1}^{\infty} \frac{F_k^n}{F_n F_{n+2} \cdots F_{n+k} F_{n+k+1}} = \frac{(F_k / F_{k+1})}{\prod_{i=1}^{k+1} F_i}, \quad k \ge 1 .$$

Solution by the Proposer.

1. $$\frac{1}{F_n F_{n+1} F_{n+2}} - \frac{1}{F_{n+1} F_{n+2} F_{n+3}} = \frac{F_{n+3} - F_n}{F_n F_{n+1} F_{n+2} F_{n+3}} = \frac{2}{F_n F_{n+2} F_{n+3}},$$

so that

$$\sum_1^{\infty} \frac{1}{F_n F_{n+2} F_{n+3} F_{n+4}} = \frac{1}{2}\sum_1^{\infty}\left(\frac{1}{F_n F_{n+1} F_{n+2} F_{n+3}} - \frac{1}{F_{n+1} F_{n+2} F_{n+3}}\right) = \frac{1}{4}.$$

2. $$\frac{1}{F_n F_{n+1} F_{n+2} F_{n+3}} - \frac{1}{F_{n+1} F_{n+2} F_{n+3} F_{n+4}} = \frac{F_{n+4} - 2F_n}{F_n F_{n+1} F_{n+2} F_{n+3} F_{n+4}}$$

$$= \frac{3}{F_nF_{n+2}F_{n+3}F_{n+4}} \quad ,$$

so that

$$3\sum_{1}^{\infty} \frac{2^n}{F_nF_{n+2}F_{n+3}F_{n+4}} = \sum_{1}^{\infty}\left(\frac{2^n}{F_nF_{n+1}F_{n+2}F_{n+3}} - \frac{2^{n+1}}{F_{n+1}F_{n+2}F_{n+3}F_{n+4}}\right)$$

$$= \frac{2}{F_1F_2F_3F_4} = \frac{1}{3} \quad .$$

3. $$\frac{1}{F_nF_{n+1}\cdots F_{n+k}} - \frac{F_k}{F_{n+1}F_{n+2}\cdots F_{n+k+1}} = \frac{F_{n+k+1} - F_kF_n}{F_nF_{n+1}\cdots F_{n+k+1}}$$

$$= \frac{F_{k+1}}{F_nF_{n+2}F_{n+3}\cdots F_{n+k+1}} \quad ,$$

so that

$$F_{k+1}\sum_{n=1}^{\infty} \frac{F_k^n}{F_nF_{n+2}F_{n+3}\cdots F_{n+k+1}} = \frac{F_k}{F_3F_4\cdots F_{k+1}}$$

which is equivalent to the stated result.

ONE MOMENT, PLEASE

H-57 *Proposed by George Ledin, Jr., San Francisco, Calif.*

If F_n is the n^{th} Fibonacci number, define

$$G_n = \left(\sum_{k=1}^{n} kF_k\right) \Bigg/ \left(\sum_{k=1}^{n} F_k\right)$$

and show

(i) $$\lim_{n\to\infty} (G_{n+1} - G_n) = 1$$

(ii) $$\lim_{n\to\infty} (G_{n+1} / G_n) = 1 \quad .$$

Generalize.

Solution by Douglas Lind, University of Virginia, Charlottesville, Va.

(i) Let H_n be the generalized Fibonacci numbers defined by $H_1 = p$, $H_2 = p + q$, $H_n = H_{n-1} + H_{n-2}$. We may show by induction

$$R_n = \sum_{k=1}^{n} kH_k = (n+1)H_{n+2} - H_{n+4} + H_3 \quad ;$$

$$S_n = \sum_{k=1}^{n} H_k = H_{n+2} - H_2$$

(The first is problem B-40, Fibonacci Quarterly, Vol. 2, No. 2, p. 154.) Let $G_n = R_n/S_n$. Then

$$L = \lim_{n\to\infty} (G_{n+1} - G_n)$$
$$= \lim_{n\to\infty} \left[\frac{(n+2)H_{n+3} - H_{n+5} + H_3}{H_{n+3} - H_2} - \frac{(n+1)H_{n+2} = H_{n+4} + H_3}{H_{n+2} - H_2} \right]$$

and so by dividing we get

$$\text{(1)} \qquad L = \lim_{n\to\infty} \left[\frac{n+2 - H_{n+5}/H_{n+3} + H_3/H_{n+3}}{1 - H_2/H_{n+3}} - \frac{n+1 - H_{n+4}/H_{n+2} + H_3/H_{n+2}}{1 - H_2/H_{n+2}} \right].$$

Horadam, "A Generalized Fibonacci Sequence," American Mathematical Monthly, Vol. 68 (1961), pp. 455-459, has shown

$$\text{(2)} \qquad \lim_{n\to\infty} H_{n+k}/H_n = a^k, \quad a = (1+\sqrt{5})/2$$

and it is easy to prove

$$\text{(3)} \qquad \lim_{n\to\infty} c/F_{n+r} = 0, \quad c, r \text{ constants},$$

so that

$$L = \lim_{n\to\infty} (n + 2 - H_{n+5}/H_{n+3} - n - 1 + H_{n+4}/H_{n+2})$$
$$= 1 - a^2 + a^2 = 1.$$

(ii) By dividing the numerator and denominator of the main fractions in (1) by n, forming their quotient, applying (2), (3), and the easily proved $\lim 1/nF_{n+r} = 0$, r constant, we find $\lim (G_{n+1}/G_n) = 1$.

Putting $p = 1$, $q = 0$ in the above gives the desired results of the problem.

Also solved by John L. Brown Jr., Penn. State Univ., State College, Pa., and the Proposer.

COMPOSITIONS ANYONE?

H-58 ***Proposed by John L. Brown Jr., Penn. State Univ., State College, Pa.***

Evaluate, as a function of n and k, the sum

$$\sum_{i_1+i_2+\cdots+i_{k+1}=n} F_{2i_1+2} F_{2i_k+2} \cdots F_{2i_k+2} F_{2i_{k+1}+2} ,$$

where $i_1, i_2, i_3, \cdots, i_{k+1}$ constitute an ordered set of indices which take on the values of all permutations of all sets of $k + 1$ non-negative integers whose sum is n.

Solution by David Zeitlin, Minneapolis, Minn.

If $V(n,k)$ is the desired sum, then

$$\sum_{n=0}^{\infty} V(n,k)x^n = \left[\sum_{n=0}^{\infty} F_{2n+2}x^n\right]^{k+1} = \frac{1}{(1 - 3x + x^2)^{k+1}} .$$

Since the generating function of the Gegenbauer (or ultraspherical) polynomial, $C_n^{(a)}(u)$, is

$$\sum_{n=0}^{\infty} C_n^{(a)}(u)x^n = \frac{1}{(1 - 2ux + x^2)^a} ,$$

we see that

$$V(n,k) = C_n^{(k+1)}\left(\frac{3}{2}\right)$$

where

$$C_n^{(a)}(\mu) = \frac{1}{\Gamma(a)} \sum_{m=0}^{[n/2]} (-1)^m \frac{\Gamma(a + n - m)}{m!(n - 2m)!} (2\mu)^{n-2m} .$$

Thus

$$V(n,k) = \sum_{m=0}^{[n/2]} (-1)^m \binom{k + n - m}{k} \binom{n - m}{m} 3^{n-2m} .$$

Also solved by the proposer.

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