SOME BINOMIAL COEFFICIENT IDENTITIES*

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1.

Put

$$H(m,n) = \sum_{i=o}^{m}\sum_{j=0}^{n}\binom{i+j}{j}\binom{m-i+j}{j}\binom{i+n-j}{n-j}\binom{m+n-i-j}{n-j}.$$

The formula

$$H(m,n) - H(m-1,n) - H(m,n-1) = \binom{m+n}{m}^2 \tag{1}$$

was proposed as a problem by Paul Brock in the SIAM Review [1]; the published solution by David Slepian established the identity by means of contour integration. Another proof was subsequently given by R. M. Baer and the proposer [2].

The writer [3] gave a proof of (1) and of some related formulas by means of generating functions. The proof of (1) in particular depended on the expansion

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\binom{i+j}{j}\binom{j+k}{k}\binom{k+\ell}{\ell}\binom{\ell+i}{i} u^i v^j w^k x^\ell \tag{2}$$

$$= \left\{[(1-v)(1-x) - w + u(1-w)]^2 - 4u(1-v-w)(1-w-x)\right\}^{-(1/2)}$$

If we take $u = w$, $v = x$ we get

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$$(3)\qquad \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} H(m,n)u^m v^n = (1 - u - v)^{-1}(1 - 2u - 2v + u^2 - 2uv + v^2)^{-(1/2)},$$

which implies (1). We now take $u = -w$, $v = -x$. Then the left member of (2) becomes

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} (-1)^{i-j}\binom{i+j}{j}\binom{j+k}{k}\binom{k+\ell}{\ell}\binom{\ell+i}{i} w^{i+k} x^{j+\ell}$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \overline{H}(m,n)\, w^m x^n ,$$

where

$$\overline{H}(m,n) = \sum_{i=0}^{m}\sum_{j=0}^{n} (-1)^{i+j}\binom{i+j}{j}\binom{m-i+j}{j}\binom{i+n-j}{n-j}\binom{m+n-i-j}{n-j}.$$

The right member of (2) becomes

$$\left\{\left[(1-u)^2 - x^2\right]^2 + 4w(1-u+x)(1-u-x)\right\}^{-\frac{1}{2}} = (1 - 2w^2 - 2x^2 + w^4 - 2w^2x^2 + x^4)^{-\frac{1}{2}}$$

It is proved in [3] that

$$(1 - 2w - 2x + w^2 - 2wx + x^2)^{-(1/2)} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{m+n}{m}^2 w^m x^n .$$

We therefore get

$$(4)\qquad \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \overline{H}(m,n)\, w^m x^n = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{m+n}{m}^2 w^{2m} x^{2n} ,$$

so that $\overline{H}(m,n) = 0$ if either m or n is odd, while

$$\overline{H}(2m,2n) = \binom{m+n}{m}^2 . \tag{5}$$

2.

If in (2) we take $u = v$, $w = x$, it is proved in [3] that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} J(m,n)v^m x^n = (1-2v)^{-(1/2)}(1-2x)^{-(1/2)}(1-2v-2x)^{-(1/2)} , \tag{6}$$

where

$$J(m,n) = \sum_{i=0}^{m}\sum_{k=0}^{n}\binom{m}{i}\binom{n}{k}\binom{m-i+k}{k}\binom{i+n-k}{i} .$$

Since

$$(1-2v)^{-(1/2)}(1-2x)^{-(1/2)}(1-2v-2x)^{-(1/2)} = (1-2v)^{-1}(1-2x)^{-1}\left\{1 - \frac{4vx}{(1-2v)(1-2x)}\right\}^{-(1/2)}$$

$$= \sum_{r=0}^{\infty}\binom{2r}{r}\frac{v^r x^r}{(1-2v)^{r+1}(1-2x)^{r+1}}$$

$$= \sum_{r=0}^{\infty}\binom{2r}{r} v^r x^r \sum_{m=0}^{\infty}\binom{m+r}{r}(2v)^m \sum_{n=0}^{\infty}\binom{n+r}{r}(2x)^n$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} 2^{m+n} v^m x^n \sum_{r=0}^{\min(m,n)} 2^{-2r}\binom{2r}{r}\binom{m}{r}\binom{n}{r} .$$

so that

$$J(m,n) = 2^{m+n} \sum_{r=0}^{\min(m,n)} 2^{-2r} \binom{2r}{r}\binom{m}{r}\binom{n}{r}$$

$$= 2^{m+n} {}_3F_2\left[\begin{matrix} 1/2, & -m, & -n \\ & 1, & 1 \end{matrix}\right]$$

in the usual notation for generalized hypergeometric function. This may be compared with [3, (4.3)].

We now take $u = -v$, $w = -x$ in (2). Then the left member of (2) becomes

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \bar{J}(m,n) v^m x^n \quad ,$$

where

$$\bar{J}(m,n) = \sum_{i=0}^{m} \sum_{k=0}^{n} (-1)^{i+k} \binom{m}{i}\binom{n}{k}\binom{m-i+k}{k}\binom{i+n-k}{i}$$

As for the right member of (2) we get

$$\left\{(1-2v)^2 + 4v(1-v+x)\right\}^{-(1/2)} = (1+4vx)^{-(1/2)} \quad ,$$

so that

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \bar{J}(m,n) v^m x^n = (1+4vx)^{-(1/2)}$$

Since

$$(1 + 4vx)^{-(1/2)} = \sum_{n=0}^{\infty} (-1)^n \binom{2n}{n} v^n x^n \quad ,$$

it follows that

$$\text{(9)} \qquad \bar{J}(m,n) = (-1)^n \binom{m+n}{m} \delta_{mn} \quad .$$

It follows from (7) that

$$\bar{J}(m,n) = (-1)^n \sum_{i=0}^{m}\sum_{k=0}^{n} (-1)^{i+k} \binom{m}{i}\binom{n}{k}\binom{i+k}{k}\binom{m+n-i-k}{n-k}$$

$$= (-1)^n \sum_{i=0}^{m}\sum_{k=0}^{n} (-1)^{i+k} \binom{m}{i}^2\binom{n}{k}^2 \frac{(i+k)!\,(m+n-i-k)}{m!\,n!}$$

Thus (9) may be replaced by

$$\text{(10)} \qquad \sum_{i=0}^{m}\sum_{k=0}^{n} (-1)^{i+k} \frac{\binom{m}{i}^2\binom{n}{k}^2}{\binom{m+n}{i+k}} = \delta_{mn} \quad .$$

3.

The left member of (3) is equal to

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \binom{i+j}{j}\binom{j+k}{k}\binom{k+\ell}{\ell}\binom{\ell+i}{i} u^{i+k} v^{j+\ell}$$

$$= \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} u^{i+k} \sum_{j=0}^{\infty} \binom{i+j}{j}\binom{k+j}{j} v^j \sum_{\ell=0}^{\infty} \binom{i+\ell}{\ell}\binom{k+\ell}{\ell} v^\ell$$

$$= \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} u^{i+k} \left\{ \sum_{j=0}^{\infty} \frac{(i+1)_j (k+1)_j}{j!\,j!} v^j \right\}^2$$

$$= \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} u^{i+k} \left\{ F(i+1,\ k+1;\ 1;\ v) \right\}^2 \quad ,$$

where $F(i+1),\ k+1;\ 1;\ v)$ is the hypergeometric function. If we put

$$G_m(v) = \sum_{k=0}^{m} \left\{ F(m-k+1,\ k+1;\ 1;\ v) \right\}^2$$

then (3) becomes

$$(11) \qquad \sum_{n=0}^{\infty} u^m G_m(v) = (1-u-v)^{-1} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \binom{m+n}{m}^2 u^m v^n \quad .$$

Multiplying by $1-u-v$ and comparing coefficients of u^m we get

$$(12) \quad (1-v)G_m(v) - G_{m-1}(v) = \sum_{n=0}^{\infty} \binom{m+n}{m}^2 v^n = F(m+1,\ m+1;1;v) \quad .$$

This identity is evidently equivalent to (1).

In a similar manner, it follows from (4) that

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} (-1)^i u^{i+k} F(i+1,\ k+1;\ 1;\ v)\ F(i+1,\ k+1;\ 1;\ -v)$$

$$= \sum_{m=0}^{\infty} u^{2m} \sum_{n=0}^{\infty} \binom{m+n}{m}^2 v^{2n} \quad ,$$

which yields the identity

$$(13)\qquad \sum_{i=0}^{2m} (-1)^i F(i+1,\, 2m-i+1,\, 1;\, v) F(i+1, 2m-i+1; 1; -v) = \sum_{n=0} \binom{m+n}{n}^2 v^{2n}.$$

The identities corresponding to (7) and (9) seem less interesting.

4.

With a little manipulation the right member of (2) reduces to

$$\left\{(1 - u - v - w - x - uw - vx)^2 - 4uvwx\right\}^{-(1/2)}$$

We have therefore

$$(14)\qquad \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \binom{i+j}{j}\binom{j+k}{k}\binom{k+\ell}{\ell}\binom{\ell+i}{i} u^i v^j w^k x^\ell$$
$$= \left\{(1 - u - v - w - x + uw + vx)^2 - 4uvwx\right\}^{-(1/2)} .$$

Note that the right side is unchanged by the permutation $(uvwx)$ and also by each of the transpositions (uw) and (vx) and therefore by the permutations of a group of order eight. The same symmetries are evident from the left member.

It may be of interest to remark that in the case of three variables we have the expansion

$$(15)\qquad \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \binom{i+j}{j}\binom{j+k}{k}\binom{k+i}{i} u^i v^j w^k$$
$$= \left\{(1 - u - v - w)^2 - 4uvw\right\}^{-(1/2)} .$$

Each side is plainly symmetric in u, v, w. As a special case of (15) we may mention $v = \epsilon u$, $w = \epsilon^2 u$, where ϵ, ϵ^2 are the primitive cube roots of unity.

The right member reduces to $(1 - 4u^3)^{-(1/2)}$ and therefore

$$\sum_{i+j+k=3n} \binom{i+j}{j}\binom{j+k}{k}\binom{k+i}{i} \epsilon^{j+2k} = \binom{2n}{n}$$

while

$$\sum_{i+j+k=n} \binom{i+j}{j}\binom{j+k}{k}\binom{k+i}{i} \epsilon^{j+2k} = 0 \quad (3 \nmid n).$$

If we expand the right member of (15) and compare coefficients we get

$$\sum_r \binom{2r}{r} \frac{(i+j+k-2r)!}{r!\,(i-r)!\,(j-r)!\,(k-r)!} = \binom{i+j}{j}\binom{j+k}{k}\binom{k+i}{i},$$

which can also be written in the form

$$(16) \qquad \sum_r \frac{\binom{i}{r}\binom{j}{r}\binom{k}{r}}{\binom{i+j+k}{2r}} = \frac{(i+j)!\,(j+k)!\,(k+i)!}{i!\,j!\,k!\,(i+j+k)!}$$

5.

In the case of six variables a good deal of computation is required. Making use of 3, (5.1) we can show that

$$(17) \qquad \sum_{i_1,\cdots,i_6=0}^{\infty} \binom{i_1+i_2}{i_2}\binom{i_2+i_3}{i_3}\binom{i_3+i_4}{i_4}\binom{i_4+i_5}{i_5}\binom{i_5+i_6}{i_6}\binom{i_6+i_1}{i_1} \cdot u_1^{i_1}u_2^{i_2}u_3^{i_3}u_4^{i_4}u_5^{i_5}u_6^{i_6}$$

$$= \{[1-u_1-u_2-u_3-u_4-u_5-u_6+u_1u_3+u_1u_4+u_1u_5+u_2u_4+u_2u_5+u_2u_6+u_3u_5+u_3u_6+u_4u_6-u_1u_3u_5-u_2u_4u_6]^2 - 4u_1u_2u_3u_4u_5u_6\}^{-\frac{1}{2}}$$

On the right of (17) the bilinear terms satisfy the following rule: in the cycle (123456) adjacent subscripts are not allowed; thus, for example u_1u_2 and u_1u_6 do not appear.

If we take $u_1 = u_4$, $u_2 = u_5$, $u_3 = u_6$, the right member of (7) reduces to

$$\left\{[1 - 2u_1 - 2u_2 - 2u_3 + (u_1 + u_2 + u_3)^2 - 2u_1u_2u_3]^2 - 4u_1^2u_2^2u_3^2\right\}^{-(1/2)}$$

$$=\{[(1 - u_1 - u_2 - u_3)^{-1}[(1 - u_1 - u_2 - u_3)^2 - 4u_1u_2u_3]^{-(1/2)}$$

in agreement with [3, (5.2)].

For five variables we find that

$$\sum_{i_1,\cdots,i_5=0} \binom{i_1+i_2}{i_2}\binom{i_2+i_3}{i_3}\binom{i_3+i_4}{i_4}\binom{i_4+i_5}{i_5}\binom{i_5+i_1}{i_1} u_1^{i_1}u_2^{i_2}u_3^{i_3}u_4^{i_4}u_5^{i_5} \tag{18}$$

$$=\left\{[1 - u_1 - u_2 - u_3 - u_4 - u_5 + u_1u_3 + u_1u_4 + u_2u_4 + u_2u_5 + u_3u_5]^2 - 4u_1u_2u_3u_4u_5\right\}^{-(1/2)}$$

The bilinear terms on the right are determined exactly as in (17); in the cycle (12345) adjacent subscripts are not allowed.

REFERENCES

1. Problem 60-2, SIAM Review, Vol. 4 (1962), pp. 396-398.
2. R. M. Baer and Paul Brock, "Natural Sorting," Journal of SIAM, Vol. 10 (1962), pp. 284-304.
3. L. Carlitz, "A Binomial Identity Arising from a Sorting Problem," SIAM Review, Vol. 6 (1964), pp. 20-30.

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