REPRESENTATIONS OF N AS A SUM OF DISTINCT ELEMENTS FROM SPECIAL SEQUENCES

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1. INTRODUCTION

Let $\{a_k\}$ denote a sequence of natural numbers which satisfies the difference equation $a_{k+2} = a_{k+1} + a_k$ for $k = 1,2,\cdots$. It is easy to prove by induction that $a_1 + a_2 + \cdots + a_n = a_{n+2} - a_2$ for $n = 1,2,\cdots$; we use this fact in defining

$$(1) \qquad P(x) = \prod_{k=1}^{\infty}\left(1 + x^{a_k}\right) = \sum_{k=0}^{\infty} A(k)\,x^k$$

and

$$(2) \qquad P_n(x) = \prod_{k=1}^{n}\left(1 + x^{a_k}\right) = \sum_{k=0}^{a_{n+2}-a_2} A_n(k)\,x^k \;.$$

It follows from these definitions that $A(k)$ enumerates the number of representations

$$(3) \qquad a_{i_1} + a_{i_2} + \cdots + a_{i_j} = k \qquad \text{with} \qquad 0 < i_1 < \cdots < i_j \;,$$

and that $A_n(k)$ enumerates the number of these representations with $i_j \le n$.

Hoggatt and Basin [9] found recurrence formulae satisfied by $\{A_n(k)\}$ and $\{A(k)\}$ when $\{a_n\}$ is the Fibonacci sequence; in Section 2 we give generalizations of these results.

Hoggatt and King [10] defined a <u>complete sequence</u> of natural numbers $\{a_n\}$ as one for which $A(n) > 0$ for $n = 1,2,\cdots$ and found that (i) $\{F_n\}$ is complete, (ii) $\{F_n\}$ with any term deleted is complete, and (iii) $\{F_n\}$ with any two terms deleted is not complete. Brown [1] gave a simple necessary and sufficient condition for completeness of an arbitrary sequence of natural numbers and showed that the Fibonacci sequence is characterized by properties (ii) and (iii) already mentioned. Zeckendorf [13] showed that if F_1 is deleted from the Fibonacci sequence, then the resulting sequence has the property that every natural number has exactly one representation as a sum of elements from this sequence whose subscripts differ by at least two. Brown [2] has given an exposition of this paper and Daykin [4] showed that the Fibonacci sequence is the only sequence with the properties mentioned in Zeckendorf's Theorem. More on the subject of Zeckendorf's Theorem can be found in another excellent paper by Brown [3]. Ferns [5], Lafer [11], and Lafer and Long [12] have discussed various aspects of the problem of representing numbers as sums of Fibonacci numbers. Graham [6] has investigated completeness properties of $\{F_n + (-1)^n\}$ and proved that every sufficiently large number is a sum of distinct elements of this sequence even after any finite subset has been deleted.

In Section 3 we take up the problem of determining the magnitude of A(n) when $\{a_n\}$ is the Fibonacci sequence; in this case we write $A(n) = R(n)$. Hoggatt [7] proposed that it be shown that $R(F_{2n} - 1) = n$ and that $R(N) > n$ if $N > F_{2n} - 1$. We will show that

$$R(F_n - 1) = \left[\frac{n+1}{2}\right] ,$$

and that $F_n \le N \le F_{n+1} - 1$ implies

$$\left[\frac{n+2}{2}\right] \le R(N) \le 2F_{(n+1)/2}$$

if n is odd and

$$\left[\frac{n+2}{2}\right] \le R(N) \le F_{(n+4)/2}$$

if n is even.

In Section 4 we investigate the number of representations of k as a sum of distinct Fibonacci numbers, writing $a_n = F_{n+1}$ and T(n) for A(n) in this case. The behavior of the function T(n) is somewhat different from that of

$R(n)$ of Section 3. For example, we show that there exist infinitely many n for which $T(n) = k$ for a fixed k, and in particular we find the solution sets for each of the equations $T(n) = 1$, $T(n) = 2$, $T(n) = 3$. By definition $T(n) \leq R(n)$ so that $T(N) \leq n - 1$ if $N \leq F_{n+1} - 1$. We show that

$$T(F_{n+1}) = \left[\frac{n+1}{2}\right]$$

and $T(F_{n+1} + 1) = [n/2]$ for $n = 3, 4, \cdots$.

Hoggatt [8] proposed that one show that $M(n)$, the number of natural numbers less than n which cannot be expressed as a sum of distinct Lucas numbers $L_n (L_1 = 1,\ L_2 = 3,\ L_{n+2} = L_{n+1} + L_n)$ has the property $M(L_n) = F_{n-1}$; also, he asked for a formula for $M(n)$. In Section 5, we give a solution to the same question involving any incomplete sequence satisfying $a_{n+2} = a_{n+1} + a_n$ with $a_1 < a_2 < \cdots$. In a paper now in preparation we have shown that the only complete sequences of natural numbers which satisfy the Fibonacci recurrence are those with initial terms (i) $a_1 = a_2 = 1$, (ii) $a_1 = 1$, $a_2 = 2$, or (iii) $a_1 = 2$, $a_2 = 1$.

2. RECURRENCE RELATIONS

See Section 1 for definitions and notation.

Lemma 1. $A_n(k) = A_n(a_{n+2} - a_2 - k)$ for $k = 0, 1, \cdots, n$.

Proof. Using the product notation for P_n we see

$$x^{a_{n+2}-a_2} P_n\left(\frac{1}{x}\right) = P_n(x) \tag{4}$$

The symmetric property of A_n now follows on equating coefficients of the powers of x in (4).

Lemma 2.

(a) $A_{n+1}(k) = A_n(k)$ if $0 \leq k \leq a_{n+1} - 1$.

(b) $A_{n+1}(k) = A_n(k) + A_n(k - a_{n+1})$ if $a_{n+1} \leq k \leq a_{n+2} - a_2$.

(c) $A_{n+1}(k) = A_n(k - a_{n+1})$ if $a_{n+2} - a_2 + 1 \leq k \leq a_{n+3} - a_2$.

Proof. Each of these statements is obtained by equating coefficients of x^k in the identity

$$P_{n+1}(x) = \left(1 + x^{a_{n+1}}\right) P_n(x) \quad . \tag{5}$$

Lemma 3.

(a) $A_{n+1}(k) = A(k)$ if $0 \leq k \leq a_{n+2} - 1$.

(b) $A_{n+1}(k) = A(a_{n+2} - a_2 - k) + A(k - a_{n+1})$ if $a_{n+1} \leq k \leq a_{n+2} - a_2$.

(c) $A_{n+1}(k) = A(a_{n+3} - a_2 - k)$ if $a_{n+2} - a_2 + 1 \leq k \leq a_{n+3} - a_2$.

Proof. (a) This follows by induction on part (a) of Lemma 2.

(c) Using Lemma 1 we have $A_{n+1}(k) = A_{n+1}(a_{n+3} - a_2 - k)$ and assuming $a_{n+2} - a_2 + 1 \leq k \leq a_{n+3} - a_2$ we have $0 \leq a_{n+3} - a_2 - k \leq a_{n+1} - 1$, so that we can apply (a) of this lemma to get $A_{n+1}(a_{n+3} - a_2 - k) = A(a_{n+3} - a_2 - k)$ for k in the range under consideration and this is (c).

(b) Statement (b) of Lemma 2 asserts $A_{n+1}(k) = A_n(k) + A_n(k - a_{n+1})$ for $A_{n+1} \leq k \leq a_{n+2} - a_2$; but by (c) of this lemma we have $A_n(k) = A(a_{n+2} - a_2 - k)$ for k in the range under consideration. Also, if $a_{n+1} \leq k \leq a_{n+2} - a_2$ we have $0 \leq k - a_{n+1} - a_2$, so by (a) of this lemma we have $A_n(k - a_{n+1}) = A(k - a_{n+1})$. Combining these results gives part b.

Lemma 4.

(a) $A(k) = A(a_{n+2} - a_2 - k) + A(k - a_{n+1})$ if $a_{n+1} \leq k \leq a_{n+2} - a_2$ and $n = 2, 3, \cdots$.

(b) $A(k) = A(a_{n+3} - a_2 - k)$ if $a_{n+2} - a_2 + 1 \leq k < a_{n+2} - 1$ and $n = 2, 3, \cdots$.

(c) $A(a_{n+2} - a_2 + k) = A(a_n - a_2 + k)$ if $1 \leq k \leq a_2 - 1$.

(Note that in (b) and (c) the range of k is the empty set unless $a_2 \geq 2$.)

Proof.

(a) This is merely a combination of (a) and (b) in Lemma 3.

(b) If $a_{n+2} - a_2 + 1 \leq k \leq a_{n+2} - 1$, then $a_{n+1} - a_2 + 1 \leq a_{n+3} - a_2 - k \leq a_{n+1} - 1$, so that by (a) of Lemma 3, $A(a_{n+3} - a_2 - k) = A_{n+1}(a_{n+3} - a_2 - k)$. By Lemma 1, $A_{n+1}(a_{n+3} - a_2 - k) = A_{n+1}(k)$ and using (a) of Lemma 3 again we see that $A_{n+1}(k) = A(k)$ for k in the proposed range.

(c) Writing $k = a_{n+2} - a_2 + j$ with $1 \leq j \leq a_2 - 1$ in (b) we get

$$A(a_{n+2} - a_2 + j) = A(a_{n+3} - a_2 - a_{n+2} + a_2 - j) = A(a_{n+1} - j) \ ; \tag{6}$$

but $a_{n+1} - j = a_{n+1} - a_2 + (a_2 - j)$ where $1 \leq a_2 - j \leq a_2 - 1$ so that we can use (6) to obtain

$$(7) \qquad A(a_{n+1} - j) = A(a_{n+1} - a_2 + (a_2 - j)) = A(a_n - (a_2 - j)) \quad .$$

Combining (6) and (7) we obtain (c).

Lemma 5. $A(a_{n+1} + j) = A(a_{n+2} - a_2 - j)$ for $0 \leq j \leq a_n - a_2$ and $n = 2, 3, \cdots$.

Proof. For j in the range under consideration we have $a_{n+1} \leq a_{n+1} + j \leq a_{n+2} - a_2$ so that by (a) of Lemma 4 we have

$$(8) \qquad \begin{aligned} A(a_{n+1} + j) &= A(a_{n+2} - a_2 - a_{n+1} - j) + A(a_{n+1} + j - a_{n+1}) \\ &= A(a_n - a_2 - j) + A(j) \quad . \end{aligned}$$

But we also have $a_{n+1} \leq a_{n+2} - a_2 - j \leq a_{n+2} - a_2$ for the assumed range of j, so that we can apply Lemma 4 again to write

$$(9) \qquad \begin{aligned} A(a_{n+2} - a_2 - j) &= A(a_{n+2} - a_2 - a_{n+2} + a_2 + j) + A(a_{n+2} - a_2 - j \\ &\quad - a_{n+1}) = A(j) + A(a_n - a_2 - j) \quad . \end{aligned}$$

Since the right members of (8) and (9) are the same, so are the left members.

Using Lemmas 4 and 5 it is not hard to calculate $A(k)$ for a given sequence $\{a_n\}$. Of particular interest to us are the cases when $\{a_n\}$ is the Fibonacci sequence, the Fibonacci sequence with the first term deleted, and the Lucas sequence; we write $A(k) = R(k)$, $T(k)$ and $S(k)$ respectively in these cases. A table is provided for each of these functions in order to illustrate some of our results.

3. SOME PROPERTIES OF R(k)

In light of Lemma 4, it is natural to consider the behavior of $R(k)$ in the intervals $[F_n, F_{n+1} - 1]$; thus, as a matter of convenience we write

$$(10) \qquad I_n = \{R(F_n), R(F_n + 1), \cdots, R(F_{n+1} - 1)\}$$

and note that Lemma 4 implies

(11) $I_{n+1} = \{R(0) + R(F_n - 1),\ R(1) + R(F_n - 2),\ \cdots,\ R(F_n - 1) + R(0)\}.$

As we mentioned in the introduction, Hoggatt has proposed that one prove $R(F_{2n} - 1) = n$ and that $R(k) > R(F_{2n} - 1)$ if $k > F_{2n} - 1$. This problem has led us to prove a result involving special values of $R(k)$ and to find the maximum and minimum of $R(k)$ in I_n.

Theorem 1.

(a) $R(F_n) = \left[\frac{n+2}{2}\right]$ for $n > 1$,

(b) $R(F_n - 1) = \left[\frac{n+1}{2}\right]$ for $n > 0$,

(c) $R(F_n - 2) = n - 2$ for $n > 2$,

(d) $R(F_n - 3) = n - 3$ for $n > 4$.

Proof. We prove only (b) (the other proofs are analogous) which implies the first part of Hoggatt's proposal. First, we observe that (b) is true for small values of n by consulting Table 1. Next, suppose

$$R(F_t - 1) = \left[\frac{t+1}{2}\right] \text{ for } t = n \text{ and } n + 1$$

and take $k = F_{n+2} - 1$ in (a) of Lemma 4 to obtain

$$R(F_{n+2} - 1) = R(0) + R(F_n - 1) = 1 + \left[\frac{n+1}{2}\right] = \left[\frac{n+3}{2}\right] \qquad (12)$$

Thus, the assertion follows by induction on n.

Theorem 2.

$$R(F_n) = \left[\frac{n+2}{2}\right]$$

is a minimum of $R(k)$ in I_n.

Proof. We can verify the theorem for small values of n by inspection in Table 1. Suppose the theorem holds for all $n \le N - 1$. We know by Theorem 1 that

$$R(F_n) = \left[\frac{n+2}{2}\right]$$

so that we are assuming

$$(13) \quad \left[\frac{n+2}{2}\right] = R(F_n) \leqq R(k) \quad \text{for} \quad F_n \leqq k \leqq F_{n+1} - 1 \quad \text{and} \quad n = 1, 2, \cdots, N-1.$$

Now suppose $F_N \leqq k \leqq F_{N+1} - 1$ and write $n = N - 1$ in (a) of Lemma 4 to obtain

$$(14) \qquad R(k) = R(F_{N+1} - 1 - k) + R(k - F_N) \ ;$$

but $F_N \leqq k \leqq F_{N+1} - 1$ implies $0 \leqq F_{N+1} - 1 - k \leqq F_{N-1} - 1$ and $0 \leqq k - F_N \le F_{N-1} - 1$. Suppose

$$(15) \qquad F_t \le F_{N+1} - 1 - k \le F_{t+1} - 1 \ ,$$

where of course $F_{t+1} - 1 \le F_{N-1} - 1$ or $0 \le t \le N - 2$ (we are taking $F_0 = 0$). Now

$$(16) \qquad F_N - F_{t+1} \le k + F_N - F_{N-1} \le F_N - F_t - 1 \ ,$$

but with $0 \le t \le N - 2$ we must have $F_{N-2} \le F_N - F_{t+1}$ and $F_N - F_t - 1 \le F_{N-1} - 1$ so that evidently

$$(17) \qquad F_{N-2} \le k - F_N \le F_{N-1} - 1 \ .$$

Using (16) and (17) along with (13) we have

$$(18) \qquad \left[\frac{N}{2}\right] \le R(k - F_N)$$

and

(19) $$1 \le \left[\frac{t+2}{2}\right] \le R(F_{N+1} - 1 - k) \text{ since } t \ge 0 .$$

Combining (18) and (19) in (14) gives

(20) $$R(k) \ge \left[\frac{N}{2}\right] + 1 = \left[\frac{N+2}{2}\right]$$

for $F_N \le k \le F_{N+1} - 1$. Hence the theorem follows by induction on N.

Corollary. $R(k) > R(F_{2n} - 1) = n$ if $k > F_{2n} - 1$.

Proof. We know from Theorem 2 that the minimum value of $R(k)$ in I_{2n} and I_{2n+1} is $n+1$ in each of them; hence the minimum of $R(k)$ in $I_{2n} \cup I_{2n+1}$ is $n+1$. Thus, every value of $R(k)$ in $I_{2n+2} \cup I_{2n+3}$ is at least $n+2$ so that we can conclude by induction on n that $R(k) > R(F_{2n} - 1)$ if $k > F_{2n} - 1$.

Theorem 3. The maximum of $R(k)$ in I_{2n} is F_{n+2} and the maximum of $R(k)$ in I_{2n+1} is $2F_{n+1}$ for $n = 1, 2, \cdots$; also,

(21) $$F_3 \le 2F_2 < F_4 < 2F_3 < \cdots < F_{n+2} < 2F_{n+1} < F_{n+3} < \cdots \quad \text{for } n = 2, 3, \cdots .$$

Proof. The result in (21) follows by a simple induction.

The results concerning the maximum values of $R(k)$ in I_{2n} and I_{2n+1} can be verified for small n by using Table 1. Suppose these results hold for all $n \le N$; then we have by (a) of Lemma 4,

(22) $$R(F_{n+1} + t) = R(F_n - t - 1) + R(t) \quad \text{for} \quad 0 \le t \le F_n - 1 .$$

Also, we know by (b) of Lemma 4 that $R(k)$ is symmetric in I_{n+1}, so it is enough to consider the values of only the first half of the elements of I_{n+1} in order to determine the maximum elements. More than the first half of the elements of I_{n+1} are contained in the sets

(23) $$\{R(F_{n+1} + t) \mid t = 0, 1, \cdots, F_{n-1} - 1\} \text{ and } \{R(F_{n+1} + t) \mid t = F_{n-1}, \cdots, F_n - 1\}.$$

Consider first the maximum of the first of the two sets in (23); evidently,

$$\begin{aligned}\max_{0\le t\le F_{n-1}-1} R(F_{n+1}+t) &= \max_{0\le t\le F_{n-1}-1}\{R(F_n - t - 1) + R(t)\}\\ &\le \max_{0\le t\le F_{n-1}-1} R(F_n - t - 1) + \max_{0\le t\le F_{n-1}-1} R(t)\\ &= 2\max I_{n-2}\ .\end{aligned}\tag{24}$$

Next, we have for the second set in (23)

$$\begin{aligned}\max_{F_{n-1}\le t\le F_n-1} R(F_{n+1}+t) &\le \max_{F_{n-1}\le t\le F_n-1} R(F_n - t - 1) + \max_{F_{n-1}\le t\le F_n-1} R(t)\\ &= \max I_{n-3} + \max I_{n-1}\ .\end{aligned}\tag{25}$$

Together (24) and (25) imply

$$\max I_{n+1} \le \max\{\max I_{n-1} + \max I_{n-3},\ 2\max I_{n-2}\}\ .\tag{26}$$

Writing $n = 2N + 1$ in (26) and applying the induction hypothesis we have

$$\max I_{2N+2} \le \max\{F_{N+3},\ 4F_N\} = F_{N+3}\ ;\tag{27}$$

similarly, $n = 2N + 2$ in (26) gives

$$\max I_{2N+3} \le \max\{2F_{N+2},\ 2F_{N+2}\} = 2F_{N+2}\ .\tag{28}$$

In order to finish the proof of Theorem 3 we need to show that $F_{N+3} \in I_{2N+2}$ and $2F_{N+2} \in I_{2N+3}$.

Since $0 \le F_{2N} + t \le F_{2N+3} - 1$ for $t = 0, 1, \cdots, F_{2N-1} - 1$, we can use (22) and (*b*) of Lemma 4 to find

$$R(F_{2N+3} + F_{2N} + t) = R(F_{2N+1} - t - 1) + R(F_{2N} + t) = 2R(F_{2N} + t)\ ,\tag{29}$$

for $t = 0, 1, \cdots, F_{2N-1} - 1$. From this we gather that all of the elements of I_{2N} multiplied by 2 occur in I_{2N+3}; hence, twice the maximum in I_{2N} is in I_{2N+3} and this is precisely $2F_{N+2}$.

It is not so obvious that $F_{N+3} \in I_{2N+2}$; to prove this we let λ_n denote an integer such that $R(F_{2N} + \lambda_n) = F_{n+2}$ for $n \leq N$. We will also include in our induction hypothesis that an admissible value of λ_{n+1} for $n < N$ is given by $\lambda_{n+1} = F_{2n-1} - \lambda_n - 1$. Now consider

$$\begin{aligned} (30) \quad & R(F_{2N+2} + F_{2N-1} - \lambda_N - 1) \\ &= R(F_{2N+1} - F_{2N-1} + \lambda_N) + R(F_{2N-1} - \lambda_N - 1) \\ &= R(F_{2N} + \lambda_N) + R(F_{2N-2} + \lambda_{N-1}) \\ &= F_{N+2} + F_{N+1} = F_{N+3} \quad . \end{aligned}$$

The second equality in (30) follows from (22). It is now clear that an admissible value for λ_{N+1} is $F_{2N-1} - \lambda_N - 1$ and that $F_{N+3} \in I_{2N+2}$. This completes the proof of Theorem 3.

4. T(n), THE NUMBER OF REPRESENTATIONS OF n AS A SUM OF DISTINCT FIBONACCI NUMBERS

For the moment we are taking $a_n = F_{n+1}$ in the lemmas of Section 2 and write $A(k) = T(k)$ in this case. The following theorem can be proved in the same way we proved Theorem 1, so we leave out the proof.

Theorem 4.

(a) $T(F_{n+1}) = \left[\frac{n+1}{2}\right]$ if $n = 1, 2, \cdots$.

(b) $T(F_{n+1} + 1) = \left[\frac{n}{2}\right]$ if $n = 3, 4, \cdots$.

Theorem 5.

(a) $T(N) = 1$ if and only if $N = F_{n+1} - 1$ for $n = 1, 2, \cdots$.

(b) $T(N) = 2$ if and only if $N = F_{n+3} + F_n - 1$ or $F_{n+4} - F_n - 1$ for $n = 1, 2, \cdots$.

(c) $T(N) > 0$ if $N \geq 0$.

(d) $T(N) = 3$ if and only if $N = F_{n+5} + F_n - 1$, $F_{n+5} + F_{n+1} - 1$, $F_{n+6} - F_n - 1$, $F_{n+6} - F_{n+1} - 1$ for $n = 1, 2, \cdots$.

Proof. (a) and (c): We can check Table 2 to see that $T(F_{n+1} - 1) = 1$ if $n = 1, 2, 3, 4$. Suppose $T(F_{n+1} - 1) = 1$ for all n less than $N > 4$. Then by (c) of Lemma 4 we have $T(F_N - F_3 + 1) = T(F_N - 1) = T(F_{N-3} - 1)$ which is 1 by assumption. Next, the table shows that the only values of $N < F_5$ for which $T(N) = 1$ are $N = F_2 - 1,\ F_3 - 1,\ F_4 - 1$ and $F_5 - 1$. Suppose for all $4 \leq n < N$, where $N > 5$, that $F_n \leq k < F_{n+1} - 1$ implies $T(k) > 1$. Then by (a) of Lemma 4 we have for $F_N \leq k < F_{N+1} - 1$, $T(k) = T(F_{N+1} - F_3 - k) + T(k - F_N) \geq 2$. This completes the proofs of both (a) and (c).

(b) By Lemma 5, we have $T(F_{n+3} + F_n - 1) = T(F_{n+4} - F_n - F_3 + 1)$, and since $F_{n+3} \leq F_{n+3} + F_n - 1 \leq F_{n+4} - F_3$ we can apply (a) of Lemma 4 to get $T(F_{n+3} + F_n - 1) = T(F_{n+4} - F_3 - F_{n+3} - F_n + 1) + T(F_{n+3} + F_n - 1 - F_{n+3})$ $= T(F_{n+1} - 1) + T(F_n - 1)$. By (a) of this lemma, the last sum is 2. To prove the "only if" part of (c), we use induction with (a) of Lemma 4 just as in the proof of the "only if" part of (a).

(d) The proof can be given using induction and (a) of Lemma 4 just as (a) and (b) were proved.

Theorem 6. For every natural number k there exist infinitely many N such that N has exactly k representations as a sum of distinct Fibonacci numbers, in fact,

$$T(F_{n+k+2} + 2F_{n+2} - 1) = k \quad \text{for } n = 1, 2, \cdots \text{ and } k = 4, 5, \cdots . \tag{31}$$

Proof. The theorem is true for $k = 1, 2, 3$, by (a), (b), and (d) of Theorem 5. We will verify the theorem for $k = 4$ and leave the verification for $k = 5$ as an exercise.

Since $F_{n+6} \leq F_{n+6} + 2F_{n+2} - 1 \leq F_{n+7} - F_3$ we can apply (a) of Lemma 4 to obtain

$$\begin{aligned} T(F_{n+6} + 2F_{n+2} - 1) &= T(F_{n+7} - F_3 - F_{n+6} - 2F_{n+2} + 1) \\ &\quad + T(F_{n+6} + 2F_{n+2} - 1 - F_{n+6}) \\ &= T(F_{n+5} - 2F_{n+2} - 1) + T(2F_{n+2} - 1)\,; \end{aligned} \tag{32}$$

however, $2F_{n+2} = F_{n+2} + F_{n+1} + F_n = F_{n+3} + F_n$ so that

$$T(2F_{n+2} - 1) = T(F_{n+3} + F_n - 1) = 2\,, \tag{33}$$

$$T(F_{n+5} - 2F_{n+2} - 1) = T(F_{n+4} - F_n - 1) = 2 \tag{34}$$

by (b) of Theorem 5; combining (33) and (34) in the last member of (32) gives the desired result.

Now suppose (31) holds for all $k < N$ where $N > 5$. Since $F_{n+N+2} \le F_{n+N+2} + 2F_{n+2} - 1 \le F_{n+N+3} - F_3$, we can use (a) of Lemma 4 to obtain

$$\begin{aligned} T(F_{n+N+2} + 2F_{n+2} - 1) &= T(F_{n+N+3} - F_3 - F_{n+N+2} - 2F_{n+2} + 1) \\ &\quad + T(F_{n+N+2} + 2F_{n+2} - 1 - F_{n+N+2}) \\ &= T(F_{n+N+1} - 2F_{n+2} - 1) + T(2F_{n+2} - 1). \end{aligned} \tag{35}$$

Since $0 < 2F_{n+2} - 1 \le F_{n+N+1} - F_3$ we can use Lemma 5 to write

$$\begin{aligned} T(F_{n+N+1} - F_3 - 2F_{n+2} + 1) &= T(F_{n+N+1} - 2F_{n+2} - 1) \\ &= T(F_{n+N} + 2F_{n+2} - 1)\ ; \end{aligned} \tag{36}$$

but, this last quantity is $n - 2$ by assumption and recalling (33) we see that the sum in the last member of (35) is $(N - 2) + 2 = N$. This concludes the proof.

5. INCOMPLETE SEQUENCES

In what follows, $N(n)$ denotes the number of non-negative integers $k \le n$ for which $A(k) = 0$.

Lemma 6. Let $0 < v_1 < v_2 < \cdots$ denote the sequence of numbers k for which $A(k) = 0$ and suppose $v_{t+1}, v_{t+2}, \cdots, v_{t+s}$ is a complete listing of the v's between a_n and $a_n + k + j \le a_{n+1}$ for $n \ge 2$; then $v_{t+j} = a_n + v_j$ for $j = 1, 2, \cdots, s$ and v_s is the largest v not exceeding $k + 1$.

Proof. The lemma can be verified for $n = 2$ and 3 by determining $A(k)$ for $0 \le k \le a_{4-1}$ using $P_4(x)$, since by (a) of Lemma 3 we have $A(k) = A_4(k)$ for k in the supposed range.

Suppose for some $N \ge 3$ that the v_i's between a_n and a_{n+1} are given by $a_n + v_1, a_n + v_2, \cdots, a_n + v_\ell$ where v_ℓ is the largest v_i not exceeding a_{n-1} and $n \le N$. We will show that this implies the v_i between a_N and $a_N + k < a_{N+1}$ are given by $a_N + v_1, a_N + v_2, \cdots, a_N + v_s$ where v_s is the largest v_i not exceeding $k + 1$.

Case 1. Let $a_N < a_N + v_j \le a_{N+1} - a_2$, then by (a) of Lemma 4 we have

$$
\begin{aligned}
A(a_N + v_j) &= A(a_{N+1} - a_2 - a_N - v_j) + A(a_N + v_j - a_N) \\
&= A(a_{N-1} - a_2 - v_j) + A(v_j) \qquad (37) \\
&= A(a_{N-1} - a_2 - v_j) \;;
\end{aligned}
$$

but for $a_N + v_j$ in the range being considered we have $0 \le v_j \le a_{N-1} - a_2$ so that by Lemma 5

$$A(a_{N-1} - a_2 - v_j) = A(a_{N-2} + v_j) \tag{38}$$

and the right member is zero by assumption, so that $A(a_N + v_j) = 0$ is a consequence.

Now suppose there is a t not a v_i such that $a_N \le a_N + t \le a_{N+1} - a_2$ and $A(a_N + t) = 0$; then by (a) of Lemma 4 we would have

$$A(a_N + t) = A(a_{N-1} - a_2 - t) + A(t) \quad . \tag{39}$$

But this is a contradiction since $A(t) \neq 0$ (t is not a v_i) and we assumed $A(a_N + t) = 0$.

Thus $a_N + v_1,\ a_N + v_2, \cdots, a_N + v_s \le a_N + k \le a_{N+1} - a_2$ is a complete listing of the v_j between a_N and $a_N + k \le a_{N+1} - a_2$.

Case 2. Let $a_{N+1} - a_2 < a_N + v_j \le a_{N+1}$, then by (c) of Lemma 4 we have

$$A(a_N + v_j) = A(a_{N-2} + v_j) \tag{40}$$

which is zero by assumption. If we suppose there is a t such that t is not a v_i and $a_{N+1} - a_2 < a_N + t < a_{N+1}$ implies $A(a_N + t) = 0$, we obtain a contradiction since $A(a_N + t) = A(a_{N-2} + t) = 0$ would imply t is a v_i.

Thus, $a_N + v_j, \cdots, a_N + v_\ell$, with v_j the smallest v_i not less than a_{N-1}, comprises a complete listing of the v_i between $a_{N+1} - a_2$ and a_{N+1}.

Taken together, the results proved in Cases 1 and 2 imply Lemma 6 by induction.

Corollary. If $A(k) > 0$ for $k \leq a_2$, then $\{a_n\}$ is complete; this is equivalent to saying $(a_1, a_2) = (2,1), (1,2)$ or $(1,1)$.

Proof. This follows from Lemma 6 and induction. Also, note that if $\{a_n\}$ is not complete, then there exist infinitely many k such that $A(k) = 0$.

Lemma 7.

(a) $N(a_n + k) = N(a_n) + N(k)$ if $0 \leq k \leq a_{n-1}$ and $n = 2, 3, 4, \cdots$.

(b) $N(k) = k$ if $0 \leq k < a_1$.

(c) $N(k) = k - 1$ if $a_1 \leq k < a_2$.

(d) $N(k) = k - 2$ if $a_2 \leq k \leq a_3$.

(e) $N(a_n - 1) = N(a_n)$ if $n = 1, 2, \cdots$.

Proof. (a) Suppose $n > 2$, then by Lemma 6, the v_i such that $a_n < v_i \leq a_n + k$ with $0 \leq k \leq a_{n-1}$ are given by $a_n + v_1, a_n + v_2, \cdots, a_n + v_j$, where v_j is the smallest v_i not exceeding k. Hence there are $N(k)$ v_i in the supposed range. By definition the number of $v_i \leq a_n$ is $N(a_n)$ so $N(a_n + k) = N(a_n) + N(k)$.

(b) (c) (d) follow from the fact that $A(k) \neq 0$ with $k < a_3$ only if $k = 0, a_1, a_2$.

(e) Since a_n is never a v_i, $N(a_n - 1) = N(a_n)$.

Lemma 8. $N(a_1) = a_1 - 1$, $N(a_2) = a_2 - 2$, $N(a_3) = a_3 - 3$ and $N(a_{n+1}) = N(a_n) + N(a_{n-1})$ if $n = 3, 4, \cdots$.

Proof. $N(a_1) = N(a_1 - 1) = a_1 - 1$ by (e) and (b) of Lemma 7 respectively; the second and third statements follow by (e) and (c) and (e) and (d) of the same lemma respectively. The last statement follows by writing $k = a_{n-1}$ in (a) of Lemma 7.

Lemma 9. $N(a_n) = a_n - F_{n+1}$ if $n = 1, 2, \cdots$ and F_n denotes the n^{th} Fibonacci number.

Proof. The statement is clearly true for $n = 1, 2, 3$ and can be seen by the first part of Lemma 8. If we suppose the statement true for all $n \leq k (k \geq 3)$ we can write

$$
\begin{aligned}
N(a_{k+1}) &= N(a_k) + N(a_{k-1}) = a_k - F_{k+1} + a_{k-1} - F_k \\
&= a_{k+1} - F_{k+2}
\end{aligned}
\tag{41}
$$

by the last part of Lemma 8; so Lemma 9 follows by induction on k.

Lemma 10. Every natural number can be written in the form

$$n = a_{k_1} + a_{k_2} + \cdots + a_{k_i} + t \tag{42}$$

with $k_j + 1 > k_{j+1}$ and $0 \leq t < a_2$.

Proof. The lemma is trivially true for all $n \leq a_2$. Every natural number between a_2 and a_3 can be written $a_2 + t$ with $t < a_1$; $n = a_3$ is of the form (42).

Suppose (42) holds for all $n < N$, and let a_k denote the largest a_i not exceeding N and consider $N - a_k$. We must have $N = a_k < a_{k-1}$, since $N < a_k \geq a_{k-1}$ implies $N \geq a_{k+1}$ which contradicts the maximal property of a_k. It follows that $N - a_k < N$ can be represented in the form (42) with $k + 1 > k_1$; hence, $N = a_k + a_{k_1} + \cdots + a_{k_i} + t$ is also of the form (42).

Theorem 7. Let n be a number represented as in (42). Then

$$N(n) = \begin{cases} n - \{F_{k_1+1} + F_{k_2+1} + \cdots + F_{k_i+1}\} & \text{if } 0 \leq t \leq a_1 \\ n - \{1 + F_{k_1+1} + F_{k_2+1} + \cdots + F_{k_i+1}\} & \text{if } a_1 \leq t \leq a_2 \end{cases} \tag{43}$$

Proof. Since $a_{k_2} + \cdots + a_{k_i} + t < a_{k_1} - 1$ we can apply Lemma 7 to obtain

$$N(n) = N(a_{k_1}) + N(a_{k_2} + \cdots + a_{k_i} + t) \; ; \tag{44}$$

applying Lemma 7 repeatedly in (44) we get

$$N(n) = N(a_{k_1}) + N(a_{k_2}) + \cdots + N(a_{k_i} + t) \; . \tag{45}$$

Now if $a_{k_i} = a_2$, $0 \leq t \leq a_1$, since if $t \geq a_1$ we would have $a_{k_i} = a_3$ and we can write

$$N(a_{k_i} + t) = N(a_{k_i}) + N(t) \; ; \tag{46}$$

but if $a_{k_i} = a_1$, we would have $a_1 + t < a_2$ and we reselect t as $a_1 + t_1$; also, we can conclude from this that $a_{k_i} - 1 \geq a_3$ so (46) still holds in this case. Thus (45) can be written in the form

$$N(n) = N(a_{k_1}) + N(a_{k_2}) + \cdots + N(a_{k_i}) + N(t) \quad . \tag{47}$$

Applying Lemma 9 to the $N(a_{k_i})$ in the right member of (47) we get

$$\begin{aligned} N(n) &= a_{k_1} - F_{k_1+1} + a_{k_2} - F_{k_2+1} + \cdots + a_{k_i} - F_{k_i+1} + N(t) \\ &= a_{k_1} + a_{k_2} + \cdots + a_{k_i} + N(t) - \{F_{k_1+1} + \cdots + F_{k_i+1}\} \quad ; \end{aligned} \tag{48}$$

but if $t < a_1$, $N(t) = t$ and $a_{k_1} + \cdots + a_{k_i} + t = n$ by assumption so that the first part of (43) is true. If $a_1 \leq t < a_2$, $N(t) = t - 1$ and we see that the second part of (43) is also true. This completes the proof of Theorem 6.

Hoggatt (Problem H-53, Fibonacci Quarterly) has proposed that one show that $M(n)$, the number of natural numbers less than n which cannot be expressed as a sum of distinct Lucas numbers $L_n (L_1 = 1,\ L_2 = 3,\ L_{n+2} = L_{n+1} + L_n$ has the property

$$M(L_n) = F_{n-1} \quad ; \tag{49}$$

also, he asked for a formula for $M(n)$.

The Lucas sequence can be used in place of $\{a_n\}$ in all of our lemmas and theorems. In particular, Lemma 9 tells us $M(L_n) = N(L_n) = L_n - F_{n+1}$; it is a trivial matter to show $L_n - F_{n+1} = F_{n-1}$ by induction so (49) is proved. Writing $a_{k_i} = L_{k_i}$ in (42) and Theorem 7 gives a formula for $M(n)$ for all natural numbers n.

Table 1
$R(k)$ for $0 \leq k < 144$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
R(n)	1	2	2	3	3	3	4	3	4	5	4	5	4	4	6	5	6	6	5	6

n	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39
R(n)	4	5	7	6	8	7	6	8	6	7	8	6	7	5	5	8	7	9	9	8

Table 1 (Cont'd)

n	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59
R(n)	10	7	8	10	8	10	8	7	10	8	9	9	7	8	5	6	9	8	11	10

n	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79
R(n)	9	12	9	11	13	10	12	9	8	12	10	12	12	10	12	8	9	12	10	13

n	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99
R(n)	11	9	12	9	10	11	8	9	6	6	10	9	12	12	11	14	10	12	15	12

n	100	101	102	103	104	105	106	107	108	109	110	111	112	113	114	115
R(n)	15	12	11	16	13	15	15	12	14	9	10	14	12	16	14	12

n	116	117	118	119	120	121	122	123	124	125	126	127	128	129	130	131
R(n)	16	12	14	16	12	14	10	9	14	12	15	15	13	16	11	12

n	132	133	134	135	136	137	138	139	140	141	142	143
R(n)	15	12	15	12	10	14	11	12	12	9	10	6

Table 2
T(k) for $0 \leq k \leq 55$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
T(n)	1	1	1	2	1	2	2	1	3	2	2	3	1	3	3	2	4	2	3	3

n	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39
T(n)	1	4	3	3	5	2	4	4	2	5	3	3	4	1	4	4	3	6	3	5

n	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55
T(n)	5	2	6	4	4	6	2	5	5	3	6	3	4	4	1	5

Table 3
S(k) for $0 \leq k \leq 68$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
S(n)	1	1	0	1	2	1	0	2	2	0	1	3	2	0	2	3	1	0	3	3

n	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39
S(n)	0	2	4	2	0	3	3	0	1	4	3	0	3	5	2	0	4	4	0	2

n	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59
S(n)	5	3	0	3	4	1	0	4	4	0	3	6	3	0	5	5	0	2	6	4

n	60	61	62	63	64	65	66	67	68	69
S(n)	0	4	6	2	0	5	5	0	3	

REFERENCES

1. J. L. Brown, Jr., "Note on Complete Sequences of Integers," American Mathematical Monthly, Vol. 67 (1960) pp. 557-560.
2. _______________, "Zeckendorf's Theorem and Some Applications," Fibonacci Quarterly, Vol. 3, No. 1, pp. 1-8.
3. _______________, "A New Characterization of the Fibonacci Numbers," Fibonacci Quarterly, Vol. 3, No. 1, pp. 1-8.
4. D. E. Daykin, "Representations of Natural Numbers as Sums of Generalized Fibonacci Numbers," Journal of the London Mathematical Society, Vol. 35 (1960) pp. 143-160.
5. H. H. Ferns, "On Representations of Integers as Sums of Distinct Fibonacci Numbers," Fibonacci Quarterly, Vol. 3, No. 1, pp. 21-30.
6. R. L. Graham, "A Property of the Fibonacci Numbers," Fibonacci Quarterly, Vol. 2, No. 1, pp. 1-10.
7. V. E. Hoggatt, Jr., Problem H-22, Fibonacci Quarterly, Vol. 2, No. 3, p. 47, also Vol. 3, No. 1, p. 45.
8. _______________, Problem H-53, Fibonacci Quarterly, Vol. 3, No. 1, p. 45.
9. V. E. Hoggatt, Jr., and S. L. Basin, "Representations by Complete Sequences, Fibonacci Quarterly, Vol. 1, No. 3, pp. 1-14.
10. V. E. Hoggatt, Jr., and C. King, Problem E 1424, American Mathematical Monthly, Vol. 67 (1960), p. 593.
11. P. Lafer, "Exploring the Fibonacci Representation of Integers," Fibonacci Quarterly, Vol. 2, No. 2, p. 114.

(Cont'd on p. 322)