ON SOME RECIPROCAL SUMS OF BROUSSEAU: AN ALTERNATIVE APPROACH TO THAT OF CARLITZ

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1. INTRODUCTION

In [2], it was shown that

$$\sum_{n=1}^{\infty}\frac{1}{F_n}=3+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{F_nF_{n+1}F_{n+2}}, \tag{1.1}$$

and, using the same approach, it can be shown that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{F_n}=1-\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+1}F_{n+2}}. \tag{1.2}$$

Let m be a positive integer, and define the sums

$$S(1,\dots,m)=\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+1}\cdots F_{n+m}}, \quad m\geq 1, \tag{1.3}$$

and

$$T(1,\dots,m)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{F_nF_{n+1}\cdots F_{n+m}}, \quad m\geq 1. \tag{1.4}$$

In [1] (see equations 20, 21, and 22), Brousseau proved that

$$T(1,2)=\frac{5}{12}-\frac{3}{2}S(1,2,3,4), \tag{1.5}$$

$$S(1,2,3,4)=\frac{97}{2640}+\frac{40}{11}T(1,2,3,4,5,6), \tag{1.6}$$

and

$$T(1,2,3,4,5,6)=\frac{589}{1900080}-\frac{273}{29}S(1,2,3,4,5,6,7,8). \tag{1.7}$$

As an application, he computed the value of the sum $\sum_{n=1}^{\infty}\frac{1}{F_n}$ to twenty-five decimal places.

Our aim in this paper is to establish explicit formulas that extend (1.5)-(1.7). Specifically, we obtain formulas

$$T(1,\dots,m)=r_1+r_2S(1,\dots,m,\ m+1,\ m+2), \quad m\geq 1, \tag{1.8}$$

and

$$S(1,\dots,m)=r_3+r_4T(1,\dots,m,\ m+1,\ m+2), \quad m\geq 1, \tag{1.9}$$

where the r_i are rational numbers that depend on m. Among other things, Carlitz [3] attacked the same problem with the use of generating functions and Fibonomial coefficients. Here we provide an alternative and more transparent approach, with the use of only simple identities that involve the Fibonacci numbers.

2. PRELIMINARY RESULTS

We require the following results:

$$\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+2}}=1; \tag{2.1}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{F_nF_{n+2}}=-1+2T(1); \tag{2.2}$$

$$\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+1}F_{n+3}}=-\frac{1}{4}+S(1,2); \tag{2.3}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{F_nF_{n+1}F_{n+3}}=\frac{1}{4}; \tag{2.4}$$

$$\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+1}\cdots F_{n+m-1}F_{n+m+1}}=\frac{(-1)^{m-1}}{F_1\cdots F_mF_{m+1}^2}+\frac{(-1)^m+F_m}{F_{m+1}}S(1,2,\ldots,m),\quad m\ge 1; \tag{2.5}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{F_nF_{n+1}\cdots F_{n+m-1}F_{n+m+1}}=\frac{(-1)^{m}}{F_1\cdots F_mF_{m+1}^2}+\frac{(-1)^{m-1}+F_m}{F_{m+1}}T(1,2,\ldots,m),\quad m\ge 1; \tag{2.6}$$

$$\begin{aligned}&F_{n+m}F_{n+m+2}-F_nF_{n+m+1}\\&=(-1)^{m-1}F_{m+1}F_{n+m+1}F_{n+m+2}+[1+(-1)^mF_{m+2}]F_{n+m}F_{n+m+2}-(-1)^{n-1}F_{m+2};\end{aligned} \tag{2.7}$$

$$\begin{aligned}&F_{n+m}F_{n+m+2}+F_nF_{n+m+1}\\&=(-1)^{m}F_{m+1}F_{n+m+1}F_{n+m+2}+[1+(-1)^{m-1}F_{m+2}]F_{n+m}F_{n+m+2}+(-1)^{n-1}F_{m+2}.\end{aligned} \tag{2.8}$$

Formulas (2.1)-(2.3) can be obtained from [1]. More precisely: (2.1) occurs as (4); (2.2) follows if we use (3) to evaluate

$$\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+k}}$$

for $k=1$ and $k=2$; and (2.3) is the first entry in Table III. Formula (2.4) follows from (2.4) in [4]. Again, turning to [4], we see that identity (3.3) therein and its counterpart for T yield (2.5) and (2.6) for $m\ge 3$. We can verify the validity of (2.5) and (2.6) for $m=1$ and 2 by simply substituting these values and comparing the outcomes with (2.1)-(2.4). Finally, (2.7) and (2.8) can be established with the use of the Binet form for F_n.

3. THE RESULTS

Our results are contained in the theorem that follows.

Theorem: Let $m\ge 1$ be an integer. Then

$$S(1,\ldots,m)=\frac{1}{(1+(-1)^{m-1}-L_{m+1})}\left[\frac{(-1)^{m-1}F_{m+2}-F_{2m+3}}{F_1\cdots F_{m+2}}-F_{m+1}F_{m+2}T(1,\ldots,m+2)\right], \tag{3.1}$$

$$T(1,\ldots,m)=\frac{1}{(1+(-1)^{m-1}+L_{m+1})}\left[\frac{(-1)^{m-1}F_{m+2}+F_{2m+3}}{F_1\cdots F_{m+2}}-F_{m+1}F_{m+2}S(1,\ldots,m+2)\right]. \tag{3.2}$$

Proof: Let $m \geq 1$ be an integer. Then, due to telescoping, we have

$$\sum_{n=1}^{\infty}\left(\frac{1}{F_n \cdots F_{n+m-1}F_{n+m+1}} - \frac{1}{F_{n+1} \cdots F_{n+m}F_{n+m+2}}\right) = \frac{1}{F_1 \cdots F_m F_{m+2}}. \tag{3.3}$$

Alternatively, with the use of (2.7), this sum can be written as

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{F_{n+m}F_{n+m+2} - F_n F_{n+m+1}}{F_n \cdots F_{n+m+2}} \\ &= \sum_{n=1}^{\infty}\frac{(-1)^{m-1}F_{m+1}F_{n+m+1}F_{n+m+2} + [1+(-1)^m F_{m+2}]F_{n+m}F_{n+m+2} - (-1)^{n-1}F_{m+2}}{F_n \cdots F_{n+m+2}} \\ &= (-1)^{m-1}F_{m+1}S(1, \ldots, m) + [1+(-1)^m F_{m+2}]\sum_{n=1}^{\infty}\frac{1}{F_n \cdots F_{n+m-1}F_{n+m+1}} - F_{m+2}\, T(1, \ldots, m+2).\end{aligned} \tag{3.4}$$

Finally, after using (2.5) to substitute for

$$\sum_{n=1}^{\infty}\frac{1}{F_n F_{n+1} \cdots F_{n+m-1}F_{n+m+1}},$$

we equate the right sides of (3.3) and (3.4), and then solve for $S(1, \ldots, m)$ to obtain (3.1). In the course of the algebraic manipulations, we make use of the well-known identities $L_n = F_{n-1} + F_{n+1}$, $F_n^2 + F_{n+1}^2 = F_{2n+1}$, and $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$.

Since the proof of (3.2) is similar, we merely give an outline. To begin, we have

$$\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{F_n \cdots F_{n+m-1}F_{n+m+1}} + \frac{1}{F_{n+1} \cdots F_{n+m}F_{n+m+2}}\right) = \frac{1}{F_1 \cdots F_m F_{m+2}}. \tag{3.5}$$

Next, we write the left side of (3.5) as

$$\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{F_{n+m}F_{n+m+2} + F_n F_{n+m+1}}{F_n \cdots F_{n+m+2}}\right). \tag{3.6}$$

Finally, we make use of (2.8) and (2.6), and then solve for $T(1, \ldots, m)$. This completes the proof of the theorem. □

If we substitute $m = 4$ into (3.1), we obtain (1.6). Likewise, if we substitute $m = 2$ and $m = 6$ into (3.2), we obtain (1.5) and (1.7), respectively.

4. CONCLUDING COMMENTS

Our results (3.1) and (3.2) do not produce (1.1) and (1.2), which, as stated in the Introduction, can be arrived at independently. Interestingly, due to our alternative approach, our main results are more simply stated than the corresponding results in [3]. See, for example, (5.8) in [3]. Incidentally, there is a typographical error in the last formula on page 464, where $-\frac{2}{3}$ should be replaced by $\frac{2}{3}$.

Finally, we refer the interested reader to the recent paper [5], where Rabinowitz discusses algorithmic aspects of certain finite reciprocal sums.

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