SOME IDENTITIES INVOLVING THE POWERS OF THE GENERALIZED FIBONACCI NUMBERS

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1. INTRODUCTION

In this paper, we are interested in the generalized Fibonacci and Lucas numbers

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n = \alpha^n + \beta^n, \tag{1}$$

where $\alpha = (p+\sqrt{p^2-4q})/2$, $\beta = (p-\sqrt{p^2-4q})/2$, and p and q are real numbers with $pq \neq 0$ and $p^2 - 4q > 0$. For $p = -q = 1$, $\{U_n\}$ and $\{V_n\}$ are the classical Fibonacci sequence $\{F_n\}$and the classical Lucas sequence $\{L_n\}$, respectively. It is obvious that the sequences $\{U_n\}$ and $\{V_n\}$ satisfy the linear recurrence relation $W_n = pW_{n-1} - qW_{n-2}$, $n \geq 2$. In [1], Zhang discussed the calculation of the summation

$$\sum_{a_1+a_2+\cdots+a_k=n} U_{a_1}U_{a_2}\ldots U_{a_k}. \tag{2}$$

This problem is very interesting and can help us to find some convolution properties. Zhang [1] gave a method for calculating (2) and obtained a series of identities involving the generalized Fibonacci numbers. For instance, he proved that

$$\sum_{a+b=n} U_aU_b = \frac{U_1}{p^2-4q}[(n-1)pU_n - 2nqU_{n-1}], \ n \geq 1, \tag{3}$$

$$\begin{aligned}\sum_{a+b+c=n} U_aU_bU_c = \frac{U_1^2}{2(p^2-4q)^2}&[((p^3-4pq)n^2 - (3p^3-6pq)n + (2p^3+4pq))U_{n-1} \\ &+ ((4q^2-p^2q)n^2 + 3p^2qn - (2p^2q+4q^2))U_{n-2}], \ n \geq 2.\end{aligned} \tag{4}$$

For the powers of the generalized Fibonacci numbers, are there results similar to (3) and (4)? It seems that this has not been studied. The purpose of this paper is to investigate the calculation of the summation of the forms

$$\sum_{a_1+a_2+\cdots+a_k=n} U_{a_1}^2U_{a_2}^2\ldots U_{a_k}^2 \quad \text{and} \quad \sum_{a_1+a_2+\cdots+a_k=n} U_{a_1}^3U_{a_2}^3\ldots U_{a_k}^3.$$

Using Zhang's method, we will establish some identities involving the squares and cubics of the generalized Fibonacci numbers.

2. MAIN RESULTS

Consider the generating functions of $\{U_n^2\}$ and $\{U_n^3\}$:

$$G(x) = \sum_{n=0}^{\infty} U_n^2x^n \quad \text{and} \quad H(x) = \sum_{n=0}^{\infty} U_n^3x^n.$$

By using (1) and the geometric series formula, we have

$$G(x)=\frac{1}{\alpha-\beta}\left[\frac{\alpha x}{(1-\alpha^2x)(1-\alpha\beta x)}-\frac{\beta x}{(1-\beta^2x)(1-\alpha\beta x)}\right],\ |x|<\min\left(\frac{1}{|\alpha^2|},\frac{1}{|\beta^2|},\frac{1}{|q|}\right),$$

and

$$H(x)=\frac{1}{(\alpha-\beta)^2}\left[\frac{U_3x}{(1-\alpha^3x)(1-\beta^3x)}-\frac{3qx}{(1-\alpha qx)(1-\beta qx)}\right],\ |x|<\min\left(\frac{1}{|\alpha^3|},\frac{1}{|\beta^3|},\frac{1}{|\alpha q|},\frac{1}{|\beta q|}\right).$$

Define

$$F(x)=\frac{G(x)}{x}=\sum_{n=1}^{\infty}U_n^2x^{n-1},\quad F_1(x)=\frac{1}{\alpha-\beta}\sum_{n=1}^{\infty}\alpha^nU_nx^{n-1},\quad F_2(x)=\frac{1}{\alpha-\beta}\sum_{n=1}^{\infty}\beta^nU_nx^{n-1}. \tag{5}$$

$$E(x)=\frac{H(x)}{x}=\sum_{n=1}^{\infty}U_n^3x^{n-1},\quad E_1(x)=\frac{1}{(\alpha-\beta)^2}\sum_{n=1}^{\infty}U_{3n}x^{n-1},\quad E_2(x)=\frac{-3}{(\alpha-\beta)^2}\sum_{n=1}^{\infty}U_nq^nx^{n-1}. \tag{6}$$

For $F(x), F_1(x), F_2(x), E(x), E_1(x)$, and $E_2(x)$, we have the following lemmas.

Lemma 1: If $F(x)$, $F_1(x)$, and $F_2(x)$ are defined by (5), then they satisfy:

$$\begin{aligned}F^2(x)=&\frac{1}{(\alpha-\beta)^3}[(p-2\alpha^2\beta x)F_1'(x)-4\alpha^2\beta F_1(x)+(p-2\alpha\beta^2x)F_2'(x)-4\alpha\beta^2F_2(x)]\\&-\frac{2\alpha\beta}{(\alpha-\beta)^4}\left[\frac{\alpha^4}{(\alpha^2-\beta^2)(1-\alpha^2x)}-\frac{\beta^4}{(\alpha^2-\beta^2)(1-\beta^2x)}-\frac{\alpha\beta}{1-\alpha\beta x}-\frac{\alpha\beta}{(1-\alpha\beta x)^2}\right];\end{aligned} \tag{7}$$

$$\begin{aligned}F^3(x)=&\frac{(p-2\alpha^2\beta x)^2F_1''(x)-14\alpha^2\beta(p-2\alpha^2\beta x)F_1'(x)+32\alpha^4\beta^2F_1(x)}{2(\alpha-\beta)^6}\\&-\frac{(p-2\alpha\beta^2x)^2F_2''(x)-14\alpha\beta^2(p-2\alpha\beta^2x)F_2'(x)+32\alpha^2\beta^4F_2(x)}{2(\alpha-\beta)^6}\\&-\frac{3\alpha\beta}{(\alpha-\beta)^6}\left[\frac{\alpha^6}{(\alpha^2-\beta^2)(1-\alpha^2x)^2}-\frac{4\alpha^6\beta}{(\alpha^2-\beta^2)(\alpha-\beta)(1-\alpha^2x)}\right.\\&-\frac{4\alpha\beta^6}{(\alpha^2-\beta^2)(\alpha-\beta)(1-\beta^2x)}+\frac{4\alpha^5\beta^2+4\alpha^2\beta^5}{(\alpha^2-\beta^2)(\alpha-\beta)(1-\alpha\beta x)}\\&\left.-\frac{\beta^6}{(\alpha^2-\beta^2)(1-\beta^2x)^2}+\frac{3\alpha^2\beta^2}{(1-\alpha\beta x)^2}+\frac{2\alpha^2\beta^2}{(1-\alpha\beta x)^3}\right].\end{aligned} \tag{8}$$

Proof: It is clear that

$$F^2(x)=F_1^2(x)+F_2^2(x)-2F_1(x)F_2(x)\quad\text{and}\quad F^3(x)=F_1^3(x)-F_2^3(x)-3F_1(x)F_2(x)F(x).$$

Using the definition of $F_1(x)$ and the derivative of $F_1(x)$, and noticing that $\alpha+\beta=p$, we get

$$F_1^2(x)=\frac{1}{(\alpha-\beta)^3}[(p-2\alpha^2\beta x)F_1'(x)-4\alpha^2\beta F_1(x)]. \tag{9}$$

Following the same pattern, we get

$$F_2^3(x)=\frac{1}{(\alpha-\beta)^3}[(p-2\alpha\beta^2x)F_2'(x)-4\alpha\beta^2F_2(x)].$$

In the meantime, it is very easy to show that

$$F_1(x)F_2(x)=\frac{\alpha\beta}{(\alpha-\beta)^4}\left[\frac{\alpha^4}{(\alpha-\beta)(1-\alpha^2x)}-\frac{\beta^4}{(\alpha^2-\beta^2)(1-\beta^2x)}-\frac{\alpha\beta}{1-\alpha\beta x}-\frac{\alpha\beta}{(1-\alpha\beta x)^2}\right],$$

where

$$|x|<\min\left(\frac{1}{|\alpha|^2},\frac{1}{|\beta|^2},\frac{1}{|q|}\right).$$

Thus, (7) holds.

Differentiating in (9), we have

$$2F_1(x)F_1'(x)=\frac{1}{(\alpha-\beta)^3}[(p-2\alpha^2\beta x)F_1''(x)-6\alpha^2\beta F_1'(x)].$$

Applying (9) again, we have

$$F_1^3(x)=\frac{(p-2\alpha^2\beta x)F_1''(x)-14\alpha^2\beta(p-2\alpha^2\beta x)F_1'(x)+32\alpha^4\beta^2F_1(x)}{2(\alpha-\beta)^6}.$$

Following the same way, we have

$$F_2^3(x)=\frac{(p-2\alpha\beta^2x)F_2''(x)-14\alpha\beta^2(p-2\alpha\beta^2x)F_2'(x)+32\alpha^2\beta^4F_2(x)}{2(\alpha-\beta)^6}.$$

On the other hand, after careful calculus, one can verify that

$$\begin{aligned}F_1(x)F_2(x)F(x)=\frac{\alpha\beta}{(\alpha-\beta)^6}\Bigg[&\frac{\alpha^6}{(\alpha^2-\beta^2)(1-\alpha^2x)^2}-\frac{4\alpha^6\beta}{(\alpha^2-\beta^2)(\alpha-\beta)(1-\alpha^2x)}\\&-\frac{4\alpha\beta^6}{(\alpha^2-\beta^2)(\alpha-\beta)(1-\beta^2x)}+\frac{4\alpha^5\beta^2+4\alpha^2\beta^5}{(\alpha^2-\beta^2)(\alpha-\beta)(1-\alpha\beta x)}\\&-\frac{\beta^6}{(\alpha^2-\beta^2)(1-\beta^2x)^2}+\frac{3\alpha^2\beta^2}{(1-\alpha\beta x)^2}+\frac{2\alpha^2\beta^2}{(1-\alpha\beta x)^3}\Bigg].\end{aligned}$$

Therefore, (8) holds. □

Lemma 2: If $E(x)$, $E_1(x)$, and $E_2(x)$ are defined by (6), then they satisfy:

$$\begin{aligned}E^2(x)=&\frac{(-2q^3x+V_3)E_1'(x)}{(p^2-4q)^2U_3}-\frac{4q^3E_1(x)}{(p^2-4q)^2U_3}+\frac{[6q^2x+3p(q+2)]E_2'(x)}{(p^2q+2p^2+4)(p^2-4q)}\\&-\frac{12qE_2(x)}{(p^2q+2p^2+4)(p^2-4q)}-\frac{6q}{p(p^2-4q)^{7/2}}\left[\frac{\alpha^6}{1-\alpha^3x}-\frac{\alpha^2q(V_2+q)}{1-\alpha qx}\right.\\&\left.+\frac{\beta^2q(V_2+q)}{1-\beta qx}-\frac{\beta^6}{1-\beta^3x}\right].\end{aligned}\tag{10}$$

$$\begin{aligned}E^3(x)=&\frac{1}{2(p^2-4q)^4U_3^2}[(V_3-2q^3x)^2E_1''(x)-14q^3(V_3-2q^3x)E_1'(x)+32q^6E_1(x)]\\&+\frac{1}{2(p^2q+2p^2+4)^2(p^2-4q)^2}\{[3p(q+2)+6q^2x]^2E_2''(x)\\&+[3p(q+2)+6q^2x](6q^2+12q)E_2'(x)+288q^2E_2(x)\}\end{aligned}$$

$$-\frac{9q}{p(p^2-4q)^5}\left\{\frac{\alpha^9}{(1-\alpha^3x)^2}+\left[\frac{\alpha^3q^3(V_2+q)}{\alpha^2-\beta^2}-\frac{\alpha^6q(V_2+q)}{\alpha-\beta}-\frac{\alpha^3q^3V_3}{\alpha^3-\beta^3}\right]\frac{1}{1-\alpha^3x}\right.$$
$$+\left(\frac{\alpha^4q^2}{\alpha-\beta}+\frac{\alpha q^4}{\alpha^2-\beta^2}\right)\frac{V_2+q}{1-\alpha qx}-\left(\frac{\beta^4q^2}{\alpha-\beta}+\frac{\beta q^4}{\alpha^2-\beta^2}\right)\frac{V_2+q}{1-\beta qx}$$
$$\left.+\left[\frac{\beta^6q(V_2+q)}{\alpha-\beta}-\frac{\beta^3q^3(V_2+q)}{\alpha^2-\beta^2}+\frac{\beta^3q^3V_3}{\alpha^3-\beta^3}\right]\frac{1}{1-\beta^3x}+\frac{\beta^9}{(1-\beta^3x)^2}\right\}+\frac{27q^2}{p(p^2-4q)^5} \tag{11}$$
$$\times\left\{\frac{\alpha^9}{(\alpha^2-\beta^2)(1-\alpha^3x)}+\left[\frac{\alpha V_1q^2(V_2+q)-\alpha^6q}{\alpha-\beta}-\frac{\beta^3q^3}{\alpha^2-\beta^2}\right]\frac{1}{1-\alpha qx}-\frac{\alpha^3q(V_2+q)}{(1-\alpha qx)^2}\right.$$
$$\left.+\left[\frac{\beta^6q-\beta V_1q^2(V_2+q)}{\alpha-\beta}+\frac{\alpha^3\beta^3}{\alpha^2-\beta^2}\right]\frac{1}{1-\beta qx}-\frac{\beta^3q(V_2+q)}{(1-\beta qx)^2}-\frac{\beta^9}{(\alpha^2-\beta^2)(1-\beta^3x)}\right\}.$$

The proof of Lemma 2 is similar to that of Lemma 1 and therefore is omitted here.

From the lemmas, we can obtain the main results of this paper. We now state and prove the following new results.

Theorem 1: Let $\{U_n\}$ be the generalized Fibonacci sequence. Then we have

$$\sum_{a+b=n} U_a^2U_b^2=\frac{[-2nqU_{n-1}+p(n-1)U_n]V_n}{p^2-4q}-\frac{2q}{p^2-4q}\left(\frac{U_{2n+2}}{U_2}-nq^{n-1}\right),\ n\geq 1, \tag{12}$$

$$\sum_{a+b+c=n} U_a^2U_b^2U_c^2=\frac{1}{2(p^2-4q)^3}[p^2n(n-1)U_{n+1}^2-2pq(2n+3)U_nU_{n+1}$$
$$+4q^2(n^2+10n-9)U_{n-1}U_{n+1}]-\frac{3q}{(p^2-4q)^3}\left[\frac{(n-1)U_{2n+2}}{U_2}\right. \tag{13}$$
$$\left.+\frac{4q(q^{n-1}V_3-V_{2n+1})}{U_2(p^2-4q)}+(n+3)(n-1)q^n\right],\ n\geq 2.$$

Proof: To show that this theorem is valid, comparing the coefficients on both sides of (7) and (8), and noticing that (1), $\alpha+\beta=p$, $\alpha\beta=q$, and $(\alpha-\beta)^2=p^2-4q$, we get identities (12) and (13). □

Corollary 1: Let $\{U_n\}$ be the generalized Fibonacci sequence and k be a positive integer. Then

$$\sum_{a+b=n} U_{ak}^2U_{bk}^2=\frac{[-2nq^kU_{nk-k}+(n-1)V_kU_{nk}]V_{nk}U_k}{p^2-4q}-\frac{2q^kU_k^2}{p^2-4q}\left(\frac{U_{2nk+2k}}{U_{2k}}-nq^{nk-k}\right),\ n\geq 1, \tag{14}$$

$$\sum_{a+b+c=n} U_{ak}^2U_{bk}^2U_{ck}^2=\frac{1}{2(p^2-4q)^3U_k^2}[n(n-1)V_k^2U_{nk+k}^2$$
$$-2V_kq^k(2n+3)U_{nk}U_{nk+k}+4q^{2k}(n^2+10n-9)U_{nk-k}U_{nk+k}] \tag{15}$$
$$-\frac{3q^k}{U_{2k}(p^2-4q)^3}\left[(n-1)U_{2nk+2k}+\frac{4(q^{nk}V_{3k}-q^kV_{2nk+k})}{U_k(p^2-4q)}+(n+3)(n-1)q^{nk}U_{2k}\right],$$

for $n\geq 2$.

Proof: Let

$$U_n' = \frac{(\alpha^k)^n - (\beta^k)^n}{\alpha^k - \beta^k} = \frac{U_{nk}}{U_k}, \quad V_n' = \alpha^{nk} + \beta^{nk} = V_{nk}. \tag{16}$$

It is clear that the sequences $\{U_n'\}$ and $\{V_n'\}$ satisfy the linear recurrence relation

$$W_n = V_k W_{n-1} - q^k W_{n-2}, \; n \geq 2.$$

Now, we apply Theorem 1 to the sequences $\{U_n'\}$ and $\{V_n'\}$. In (12), if U_n, V_n, p, and q are replaced by U_n', V_n', V_k, and q^k, respectively, one has that

$$\sum_{a+b=n} U_a'^2 U_b'^2 = \frac{[-2nq^k U_{n-1}' + V_k(n-1)U_n']V_n'}{V_k^2 - 4q^k} - \frac{2q^k}{V_k^2 - 4q^k}\left(\frac{U_{2n+2}'}{U_2'} - nq^{nk-k}\right).$$

Due to (16) and $V_k^2 - 4q^k = U_k^2(p^2 - 4q)$, we get (14). Using a similar method, we get (15). □

Theorem 2: Let $\{U_n\}$ be the generalized Fibonacci sequence. Then

$$\begin{aligned}\sum_{a+b=n} U_a^3 U_b^3 = {} & \frac{-2nq^3 U_{3(n-1)} + (n-1)V_3 U_{3n}}{(p^2-4q)^3 U_3} - \frac{9q^n(2nqU_{n-1} - pU_n)}{(p^2-4q)^3} \\ & - \frac{6q[U_{3n} - (V_2+q)q^{n-1}U_n]}{p(p^2-4q)^3},\end{aligned} \tag{17}$$

$$\begin{aligned}\sum_{a+b+c=n} U_a^3 U_b^3 U_c^3 = {} & \frac{(n-2)[(n-1)V_3^2 U_{3n} - 2(2n+1)q^3 V_3 U_{3n-3}] + 4(n^2-1)q^6 U_{3n-6}}{2(p^2-4q)^5 U_3^2} \\ & - \frac{27q^n}{(p^2-4q)^5}[4q^2(n^2-1)U_{n-2} - 2pq(7n-12)U_{n-1} + p^2(n-1)(n-2)U_n] \\ & - \frac{9q}{p(p^2-4q)^5}\left[(n-2)V_{3n} + \frac{(V_2+q)q^3 U_{3n-6}}{p} - q(V_2+q)U_{3n-3} - \frac{q^3 V_3 U_{3n-6}}{U_3}\right. \\ & \left. + (V_2+q)q^{n-1}U_{n+1} + \frac{(V_2+q)q^{n+1}U_{n-2}}{p}\right] + \frac{27q^2}{p(p^2-4q)^5}\left[\frac{U_{3n}}{p} - q^{n-2}U_{n+3}\right. \\ & \left. - V_1(V_2+q)q^{n-1}U_{n-2} - \frac{q^{n+3}U_{n-6}}{p} - (n-2)(V_2+q)q^{n+1}V_{n+1}\right].\end{aligned}$$

Proof: Comparing the coefficients of x^{n-2} and x^{n-3} on both sides of (10) and (11), respectively, we get Theorem 2. □

Corollary 2: Let $\{U_n\}$ be the generalized Fibonacci sequence and k be a positive integer. Then

$$\begin{aligned}\sum_{a+b=n} U_{ak}^3 U_{bk}^3 = {} & \frac{-2nq^{3k}U_{3k(n-1)} + (n-1)V_{3k}U_{3kn}}{U_{3k}(p^2-4q)^3} - \frac{9q^{kn}(2nq^k U_{nk-k} - V_k U_{kn}}{U_k(p^2-4q)^3} \\ & - \frac{6q^k[U_{3kn} - (V_{2k}q^{kn-k} + q^{kn})U_{nk}]}{U_k V_k (p^2-4q)^3},\end{aligned}$$

$$\sum_{a+b+c=n} U_{ak}^3 U_{bk}^3 U_{ck}^3 = \frac{1}{2U_{3k}^2(p^2-4q)^5}[(n-1)(n-2)V_{3k}^2 U_{3kn} - 2(n-2)(2n+1)q^{3k}U_{3kn-3k}V_{3k}$$
$$+4(n^2-1)q^{6k}U_{3kn-6k}] - \frac{27q^{nk}}{U_k^2(p^2-4q)^5}[4q^{2k}(n^2-1)U_{kn-2k}$$
$$-2V_k q^k(7n-12)U_{kn-k} + V_k^2(n-1)(n-2)U_{nk}] - \frac{9q^k}{U_kV_k(p^2-4q)^5}$$
$$\times\left[(n-2)V_{3kn} + \frac{q^{3k}(V_{2k}+q^k)U_{3kn-6k}}{U_kV_k} - \frac{q^k(V_{2k}+q^k)U_{3kn-3k}}{U_k}\right.$$
$$\left. - \frac{q^{3k}V_{3k}U_{3kn-6k}}{U_{3k}} + \frac{q^{kn-k}(V_{2k}+q^k)U_{kn+k}}{U_k} + \frac{q^{kn+k}(V_{2k}+q^k)U_{kn-2k}}{U_kV_k}\right]$$
$$+\frac{27q^{2k}}{U_k^2V_k(p^2-4q)^5}\left[\frac{U_{3kn}}{V_k} - q^{kn-2k}U_{kn+3k} - V_k(V_{2k}+q^k)q^{kn-k}U_{kn-2k}\right.$$
$$\left. - \frac{q^{kn+3k}U_{kn-6k}}{V_k} - (n-2)(V_{2k}+q^k)q^{kn+k}V_{kn+k}U_k\right].$$

We note that Corollaries 1 and 2 are generalizations of Theorems 1 and 2, respectively.

Finally, we can find some congruences from Theorems 1 and 2 according to the particular choices of p and q. For example, setting $p=-q=1$ in (12) and $F_2=1$, we obtain

$$\sum_{a+b=n} F_a^2F_b^2 = \frac{[2nF_{n-1}+(n-1)F_n]L_n}{5} + \frac{2}{5}(F_{2n+2}+n(-1)^n).$$

Setting $p=-q=1$ in (17), we have

$$\sum_{a+b=n} F_a^3F_b^3 = \frac{nF_{3n-3}+2(n+2)F_{3n}+21(-1)^nF_n+18(-1)^nF_{n-1}}{125}.$$

Hence,

$$[2nF_{n-1}+(n-1)F_n]L_n + 2(F_{2n+2}+n(-1)^n) \equiv 0 \pmod 5,\ n\ge 1,$$

and

$$nF_{3n-3}+2(n+2)F_{3n}+(-1)^n(21nF_n+18nF_{n-1}) \equiv 0 \pmod{125},\ n\ge 1.$$

REFERENCE

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