# ADVANCED PROBLEMS AND SOLUTIONS

*Edited by*
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*Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, IMATE, UNAM, AP. POSTAL 61-3 (XANGARI),CP 58 089, MORELIA, MICHOACAN, MEXICO, or by e-mail at fluca@matmor.unam.mx as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.*

## PROBLEMS PROPOSED IN THIS ISSUE

**H-596 Proposed by the Editor**

A beautiful result of McDaniel (*The Fibonacci Quarterly* **40.1**, 2002) says that $F_n$ has a prime divisor $p \equiv 1 \pmod 4$ for all but finitely many positive integers $n$. Show that the asymptotic density of the set of positive integers $n$ for which $F_n$ has a prime divisor $p \equiv 3 \pmod 4$ is 1/2. Recall that a subset $\mathcal{N}$ of all the positive integers is said to have an asymptotic density $\lambda$ if the limit

$$\lim_{x\to\infty} \frac{\#\{1 \leq n < x \mid n \in \mathcal{N}\}}{x}$$

exists and equals $\lambda$.

**H-597 Proposed by Mario Catalani, University of Torino, Torino, Italy**

Let $\alpha, \beta, \gamma$ be the roots of the trinomial $x^3 - x^2 - x - 1 = 0$. Express

$$U_n = \sum_{i=1}^{n} \sum_{j=0}^{n-i} \alpha^i \beta^j \gamma^{n-i-j}$$

interms of the Tribonacci sequence $\{T_n\}$ given by $T_0 = 0,\ T_1 = 1,\ T_2 = 1$ and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 3$.

**H-598 Proposed by José Díaz-Barrero & Juan Egozcue, Barcelona, Spain**

Show that all the roots of the equation

$$\begin{vmatrix} F_1F_n & \dots & F_1F_3 & F_1F_2 & F_1^2 - x \\ F_2F_n & \dots & F_2F_3 & F_2^2 - x & F_2F_1 \\ \dots & \dots & \dots & \dots & \dots \\ F_n^2 - x & \dots & F_nF_3 & F_nF_2 & F_nF_1 \end{vmatrix} = 0$$

are integers.

## SOLUTIONS

### Representing reals in Fibonacci series

**H-582 Proposed by Ernst Herrmann, Siegburg, Germany**

**a)** Let $A$ denote the set $\{2,3,5,8,\dots,F_{m+2}\}$ of $m$ succesive Fibonacci numbers, where $m \geq 4$. Prove that each real number $x$ of the interval $I = [(F_{m+2}-1)^{-1},1]$ has a series representation of the form

$$x = \sum_{i=1}^{\infty} \frac{1}{F_{k_1}F_{k_2}\dots F_{k_i}}, \tag{1}$$

where $F_{k_i} \in A$ for all $i \in N$.

**b)** It is impossible to change the assumption $m \geq 4$ into $m \geq 3$, that is, if $A = \{2,3,5\}$ and $I = [1/4,1]$, then there are real numbers with no representation of the form (1), where $F_{k_i} \in A$. Find such a number.

**Solution by Paul Bruckman, Sacramento, CA**

Given an infinite sequence $\{c_n\}$ of real numbers with $c_n \geq 2$ write $S(c_1,c_2,\dots)$ for the value of the series $1/c_1 + 1/(c_1c_2) + \dots$. Note that $S(c_1,c_2,\dots)$ is well defined and is in the interval $(0,1]$.

Now consider the series $S(F_{k_1},F_{k_2},\dots)$, where $k_i \geq 3$ for $i \in N$. For notational convenience write $U(k_1,k_2,\dots) = S(F_{k_1},F_{k_2},\dots)$. We first show that for all $x$ with $0 < x \leq 1$, $x$ has an $U$-series with no restriction of the subscripts $k_i$ other than $k_i \geq 3$ for $i \in N$. To see why this is so, observe that for all real numbers $A \geq 1$, there is always a Fibonacci number $F_j \geq 2$, such that $A < F_j \leq 2A$. In particular, given $x_1$ with $0 < x_1 \leq 1$, we may choose $k_1 \geq 3$ such that $1/x_1 < F_{k_1} \leq 2/x_1$. Let $x_2 = x_1F_{k_1} - 1$. Clearly, $0 < x_2 \leq 1$, and therefore we may repeat the above algorithm. In other words, there exists $k_2 \geq 3$ such that if we write $x_3 = x_2F_{k_2} - 1$, then $0 < x_3 \leq 1$. Continuing in this fashion, we generate an infinite sequence $\{k_1,k_2,\dots\}$ such that $x_1 = U(k_1,k_2,\dots)$. Note that, in general, such a sequence is not uniquely determined.

We now prove **a)**. Given $m \geq 4$, let $I_m = [(F_{m+2}-1)^{-1},1]$, write $A_m = \{F_3,F_4,\dots,F_{m+2}\}$, and consider a given $x_1$ in $I_m$. As we showed above, there exists a sequence $\{k_i\}$ with $k_i \geq 3$ for $i \in N$ such that $x_1 = U(k_1,k_2,\dots)$. We prove that among all such sequences there exists one which satisfies the additional constraint that $k_i \in A_m$ for all $i \in N$. To achieve this, we partition $I_m$ into $m$ disjoint intervals as follows:

a) suppose first that $(F_{m+2}-1)^{-1} \leq x_1 \leq 2/F_{m+2}$. Then $x_1 \leq x_1F_{m+2} - 1 \leq 1$. Thus, we may choose $k_1 = m+2$, put $x_2 = x_1F_{k_1} - 1$, and then $x_2 \in I_m$ and we may continue the process. Note that there might be other values of $k_1$ for which $x_2$ is in $I_m$.

b) suppose now that $2/F_{k+1} < x_1 \leq 2/F_k$ for some $k = 3,4,\dots,m+1$. Then, $F_k \leq 2/x_1$ and $F_{k+1} > 2/x_1$. Since $F_{k+1} < 2F_k$, it follows that $1/x_1 < F_k \leq 2/x_1$. Note that there might be other values of $k$ for which this last inequality is satisfied. Choose $k_1 = k$ and write $x_2 = x_1F_{k_1} - 1$. Then $x_2 \leq 1$, and

$$x_2 > \frac{2F_k}{F_{k+1}} - 1 = \frac{F_{k-2}}{F_{k+1}}.$$

Now note that the right hand side of the above inequality is larger than or equal to $(F_{m+2} - 1)^{-1}$when $m \geq 4$. Indeed, the inequality

$$\frac{F_{k-2}}{F_{k+1}} \geq \frac{1}{F_{m+2}-1}$$

is equivalent to

$$F_{k-2}F_{m+2} \geq F_{k+1} + F_{k-2}.$$

The above inequality holds for all $m \geq 4$ and $k \in \{3, 4, \ldots, m+1\}$, but fails at $m = 3$ and $k = m+1$. Thus, when $m \geq 4$, the number $x_2 \in I_m$ and we may continue the process. This proves part **a**).

For part **b**), consider the interval $J_3 = (2/5, 5/12) \subset I_3$. If we take $x_1 \in J_3$, we see that $k_1 = 4$ is the only possibility. We then obtain $x_2 = x_1 F_{k_1} - 1 = 3x_1 - 1$, hence $1/5 < x_2 < 1/4$, and it is now clear that it is not possible that $k_i \in \{3, 4, 5\}$ for all $i \geq 2$. This argument shows that *all* values of $x_1 \in J_3$ have the property that they do not have a representation of the form (1) with $k_i \in A_3$ for all $i \in N$, which, in particular, answers both questions from part **b**).

Bruckman also attaches some examples of specific representations of the form (1) for some numbers stressing on the fact that such representations are, in general, not unique. A nice one is

$$0.41 = U(4,5,6;\overline{6,3}) = U(4,5,7,3,3,3;\overline{3,6}) =$$

$$U(4,6,3,3;\overline{3,4}) = U(4,6,3,3,4,8,3;\overline{4,7,9}) = U(4,6,3,3;\overline{4,7,9}),$$

where the bar notation above has the same meaning as the one from the theory of periodic continued fractions. Note that $0.41 \in J_3$ so no such representation of it exists with $k_i \in A_3$ for all $i \in N$.

**Also solved by the proposer.**

## Identities with Fibonacci polynomials

**H-586 Proposed by H.-J. Seiffert, Berlin, Germany**

Define the sequence of Fibonacci and Lucas polynomials by

$$F_0(x) = 0,\ F_1(x) = 1,\ F_{n+1}(x) = xF_n(x) + F_{n-1}(x),\ n \in N,$$

$$L_0(x) = 2,\ L_1(x) = x,\ L_{n+1}(x) = xL_n(x) + L_{n-1}(x),\ n \in N,$$

respectively. Show that, for all complex numbers $x$ and all positive integers $n$,

$$\sum_{k=0}^{[n/2]} \binom{n-k}{k} x^k F_{3k}(x) = \frac{xF_{2n+1}(x) - F_{2n}(x) + (-x)^{n+2}F_n(x) + (-x)^{n+1}F_{n-1}(x)}{2x^2-1}$$

and

$$\sum_{k=0}^{[n/2]} \binom{n-k}{k} x^k L_{3k}(x) = \frac{xL_{2n+1}(x) - L_{2n}(x) + (-x)^{n+2}L_n(x) + (-x)^{n+1}L_{n-1}(x)}{2x^2-1}.$$

**Solution by the proposer**

It is well known that

$$F_{n+1}(x) = \frac{\alpha(x)^{n+1} - \beta(x)^{n+1}}{\sqrt{x^2+4}}, \tag{1}$$

where $\alpha(x) = (x+\sqrt{x^2+4})/2$ and $\beta(x) = (x-\sqrt{x^2+4})/2$, and that

$$F_{n+1}(x) = \sum_{k=0}^{[n/2]} \binom{n-k}{k} x^{n-2k}. \tag{2}$$

Each of the sides of the desired identities becomes a polynomial in $x$ when multiplied by $2x^2-1$. Thus, it suffices to prove these identities for real numbers $x > 1$. For such $x$, let $y = \sqrt{\alpha(x)/x} - \sqrt{-x\beta(x)}$. Since $\alpha(x)\beta(x) = -1$, we have

$$\sqrt{y^2+4} = \sqrt{\alpha(x)/x} + \sqrt{-x\beta(x)} = \frac{\alpha(x)+x}{\sqrt{x\alpha(x)}},$$

$$y + \sqrt{y^2+4} = 2\sqrt{\alpha(x)/x} \qquad \textit{and} \qquad y - \sqrt{y^2+4} = -2\sqrt{-x\beta(x)}.$$

Noting that $(\alpha(x)+x)(\beta(x)+x) = 2x^2-1$, from (1), it now easily follows that

$$x^{n/2}\alpha(x)^{3n/2}F_{n+1}(y) = \frac{\beta(x)+x}{2x^2-1} \cdot \left(\alpha(x)^{2n+1} - (-x)^{n+1}\alpha(x)^n\right),$$

or, since $\alpha(x)\beta(x) = -1$,

$$x^{n/2}\alpha(x)^{3n/2}F_{n+1}(y) = \frac{x\alpha(x)^{2n+1} - \alpha(x)^{2n} + (-x)^{n+2}\alpha(x)^n + (-x)^{n+1}\alpha(x)^{n-1}}{2x^2-1}. \tag{3}$$

From $\beta(x)^3 = (x^2+1)\beta(x)+x = x^2\beta(x)-\alpha(x)+2x$, we obtain $-\beta^3(x)/x = \alpha(x)/x - x\beta(x)-2$, giving $y = \sqrt{-\beta^3(x)/x}$.Since $\alpha(x)\beta(x) = -1$, from (2), it follows that

$$x^{n/2}\alpha(x)^{3n/2}F_{n+1}(y) = \sum_{k=0}^{[n/2]} \binom{n-k}{k} x^k \alpha(x)^{3k}. \tag{4}$$

Combining (3) and (4) and using the known relations $2\alpha(x)^j = L_j(x) + \sqrt{x^2+4}F_j(x)$, we obtain the desired identities.

**Also solved by P. Bruckman, M. Catalani, K. Davenport, and V. Mathe.**

## Matrices with Fibonacci Polynomials

**H-587 Proposed by N. Gauthier & J.R. Gosselin, Royal Military College of Canada**

Let $x$ and $y$ be indeterminates and let

$$\alpha \equiv \alpha(x,y) = \frac{1}{2}(x + \sqrt{x^2+4y}), \ \beta \equiv \beta(x,y) = \frac{1}{2}(x - \sqrt{x^2+4y})$$

be the distinct roots of the characteristic equation for the generalized Fibonacci sequence $\{H_n(x,y)\}_{n=0}^{n=\infty}$, where

$$H_{n+2}(x,y) = xH_{n+1}(x,y) + yH_n(x,y).$$

If the initial conditions are taken as $H_0(x,y) = 0$, $H_1(x,y) = 1$, then the sequence gives the generalized Fibonacci polynomials $\{F_n(x,y)\}_{n=0}^{n=\infty}$. On the other hand, if $H_0(x,y) = 2$, $H_1(x,y) = x$, then the sequence gives the generalized Lucas polynomials $\{L_n(x,y)\}_{n=0}^{n=\infty}$.

Consider the following $2 \times 2$ matrices,

$$A = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}, \quad B = \begin{pmatrix} \beta & 1 \\ 0 & \beta \end{pmatrix}, \quad C = \begin{pmatrix} \alpha & 1 \\ 0 & \beta \end{pmatrix}, \quad D = \begin{pmatrix} \beta & 1 \\ 0 & \alpha \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and let $n$ and $m$ be nonnegative integers. [By definition, a matrix raised to the power 0 is equal to the unitmatrix $I$.]

**a.** Express $f_{n,m}(x,y) \equiv [(A-B)^{-1}(A^n - B^n)]^m$ in closed form, in terms of the Fibonacci polynomials.

**b.** Express $g_{n,m}(x,y) \equiv [A^n + B^n]^m$ in closed form, in terms of the Lucas polynomials.

**c.** Express $h_{n,m}(x,y) \equiv [C^n + D^n]^m$ in closed form, in terms of the Fibonacci and Lucas polynomials.

**Combined solution by Paul Bruckman, Sacramento, CA and Mario Catalani, Torino, Italy**

To simplify notations, we write $\alpha = \alpha(x,y)$, $\beta = \beta(x,y)$, $F_n = F_n(x,y)$, and $L_n = L_n(x,y)$. The Binet formulas for the Fibonacci and Lucas polynomialsare

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n.$$

Clearly,

$$(A-B)^{-1} = \frac{I}{\alpha - \beta}.$$

By an easy induction on $n$ one provesthat

$$A^n = \begin{pmatrix} \alpha^n & n\alpha^{n-1} \\ 0 & \alpha^n \end{pmatrix}, \qquad B^n = \begin{pmatrix} \beta^n & n\beta^{n-1} \\ 0 & \beta^n \end{pmatrix}, \qquad A^n - B^n = \begin{pmatrix} \alpha^n - \beta^n & n(\alpha^{n-1} - \beta^{n-1}) \\ 0 & \alpha^n - \beta^n \end{pmatrix}$$

and

$$C^n = \begin{pmatrix} \alpha^n & F_n \\ 0 & \beta^n \end{pmatrix}, \qquad D^n = \begin{pmatrix} \beta^n & F_n \\ 0 & \alpha^n \end{pmatrix}, \; C^n + D^n = \begin{pmatrix} \alpha^n + \beta^n & 2F_n \\ 0 & \alpha^n - \beta^n \end{pmatrix}.$$

By induction on $m$ when $n$ is fixed, it now follows that

$$[(A-B)^{-1}(A^n - B^n)]^m = \begin{pmatrix} F_n & nF_{n-1} \\ 0 & F_n \end{pmatrix}^m = \begin{pmatrix} F_n^m & nmF_n^{m-1}F_{n-1} \\ 0 & F_n^m \end{pmatrix},$$

$$[A^n + B^n]^m = \begin{pmatrix} L_n & nL_{n-1} \\ 0 & L_n \end{pmatrix}^m = \begin{pmatrix} L_n^m & nmL_n^{m-1}L_{n-1} \\ 0 & L_n^m \end{pmatrix},$$

and

$$[C^n + D^n]^m = \begin{pmatrix} L_n & 2F_n \\ 0 & L_n \end{pmatrix}^m = \begin{pmatrix} L_n^m & 2mL_n^{m-1}F_n \\ 0 & L_n^m \end{pmatrix}.$$

**Also solved by the proposers.**

**Errata:** In the displayed formula in Proposed Problem H-595 (volume 41.1) the equal sign "=" should have been "≤".

**Please Send in Proposals!**