ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Russ Euler and Jawad Sadek

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by November 15, 2003. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-956 *Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI*

Prove that

$$\frac{1+\sqrt{5}}{4} \le \sum_{n=0}^{\infty} \frac{1}{L_{2n}} \le \frac{3}{2}.$$

B-957 *Proposed by Muneer Jebreel, Jerusalem, Israel*

For $n \ge 1$, prove that

(a) $L^2_{2^n+3} + 4 = 4L_{2^{n+1}+3} + L^2_{2^n}$

and

(b) $L^2_{2^n+6} = 4 + 4L_{2^{n+1}+9} + L^2_{2^n+3}$.

B-958 *Proposed by José Luis Diaz-Barrero & Juan José Egozcue, Universitat Politècnica de Catalunya, Barcelona, Spain*

Find the greatest common divisor of

$$2+\sum_{k=1}^{n} L_k^2 \text{ and } 3+\sum_{k=1}^{n} L_k.$$

B-959 *Proposed by John H. Jaroma, Austin College, Sherman, TX*

Find the sum of the infinite series

$$1+\frac{1}{2}+\frac{1}{4}+\frac{3}{8}+\frac{3}{16}+\frac{8}{32}+\frac{7}{64}+\frac{21}{128}+\frac{15}{256}+\frac{55}{512}+\frac{31}{1024}+\frac{144}{2048}+\frac{63}{4096}+\frac{377}{8192}+\frac{127}{16384}+\cdots.$$

B-960 *Proposed by Bob Johnson, Durham University, Durham, England*

If $a+b=c+d$, prove that

$$F_aF_b - F_cF_d = (-1)^r\left[F_{a-r}F_{b-r} - F_{c-r}F_{d-r}\right]$$

for all integers a, b, c, d and r.

SOLUTIONS

Circle the Squares

B-940 *Proposed by Gabriela Stănică & Pantelimon Stănică, Auborn Univ. Montgomery, Montgomery AL.*
(Vol. 40, no. 4, August 2002)

How many perfect squares are in the sequence

$$x_n = 1+\sum_{k=0}^{n} F_k! \quad \text{for } n \geq 0?$$

Solution by Martin Reiner, New York, NY.

We claim that the only square in this sequence is $x_2 = 4$.

Note that $x_0, \ldots, x_4 = 2, 3, 4, 6, 12$. For $k \geq 5$ we have $F_k \geq 5$, and so $F_k! \equiv 0 \pmod 5$. Thus $x_k \equiv x_{k-1} \equiv \cdots \equiv x_4 \equiv 2 \pmod 5$. But modulo 5 any square is either congruent to 0, 1, or 4.

Also solved by Scott Brown, Paul Bruckman, Ovidiu Furdiu, Walther Janous, Jaroslaze Seibert, H.-J. Seiffert, and the proposer.

All solutions received follow more or less the same method as the featured one.

It is Always Negative

B-941 Proposed by Walther Janous, Innsbruck, Austria (Vol. 40, no. 4, August 2002)

Show that

$$\frac{nF_{n+6}}{2^{n+1}} + \frac{F_{n+8}}{2^n} - F_8 < 0 \text{ for } n \geq 1.$$

Solution by José Luis Díaz-Barrero and Juan José Egozcue, Universitat Politècnica de Catalunya, Barcelona, Spain.

First, we observe that the given statement is equivalent to

$$nF_{n+6} + 2F_{n+8} < 2^{n+1}F_8. \tag{1}$$

In order to prove the preceding inequality we will argue by induction. The case when $n = 1$ trivially holds. Assume that (1) holds for $n = 1, 2, \ldots, n$ and we should prove that

$$(n+1)F_{n+7} + 2F_{n+9} < 2^{n+2}F_8$$

also holds. In fact,

$$\begin{aligned}(n+1)F_{n+7} + 2F_{n+9} &= (n+1)(F_{n+6} + F_{n+5}) + 2(F_{n+7} + F_{n+8}) \\ &= (nF_{n+6} + 2F_{n+8}) + [(n-1)F_{n+5} + 2F_{n+7}] + 2F_{n+5} + F_{n+6} \\ &< 2^{n+1}F_8 + 2^nF_8 + (n-1)F_{n+5} + 2(F_{n+5} + F_{n+6}) < 2^{n+2}F_8\end{aligned}$$

and the result stated in (1) follows by strong mathematical induction.

H.-J. Seiffert showed that $\frac{nF_{n+6}}{2^{n+1}} + \frac{F_{n+8}}{2^n} - F_8 \leq -\frac{3}{4}$ *for* $n \geq 1$ *and L.A.G. Dresel generalized the inequality to* $\frac{nF_{n+t}}{2^{n+1}} + \frac{2F_{n+t+2}}{2^{n+1}} - F_{n+t+2} < 0$ *for* $n \geq 1$ *and* $t \geq 3$.

The condition $n \geq 1$ *was inadvertantly left out by the editor.*

Also solved by Charles Ashbacker, Gurdial Arora and Donna Stutson, Paul Bruckman, Mario Catalani, Kenneth Davenport, L.A.G. Dresel, Ovidiu Furdui, John Jaroma, Gerald A. Heuer, Jaroslav Seibert, H.J. Seiffert, Adam Stinchcomb, and the proposer.

As Close As It Gets

B-942 Proposed by Stanley Rabinowitz, MathPro Press, Westford, MA (Vol. 40, no. 4, August 2002)

(a) For $n > 3$, find the Fibonacci number closest to L_n.

(b) For $n > 3$, find the Fibonacci number closest to L_n^2.

Solution by L.A.G. Dresel, Reading England.

(a) The identity $L_n = F_{n+1} + F_{n-1}$ gives $L_n = (F_{n+1} + F_n) + (F_{n-1} - F_n) = F_{n+2} - F_{n-2}$. Therefore $F_{n+1} < L_n < F_{n+2}$, and as we have $F_{n-2} < F_{n-1}$ for $n \geq 4$, it follows that F_{n+2} is the Fibonacci number closest to L_n.

(b) Since $L_n = \alpha^n + \beta^n$ we have $(L_n)^2 = L_{2n} + 2(-1)^n$. As before, we have

$$L_{2n} = F_{2n+1} + F_{2n-1} = F_{2n+2} - F_{2n-2}, \text{ so that}$$

$$(L_n)^2 = F_{2n+1} + F_{2n-1} + 2(-1)^n = F_{2n+2} - \{F_{2n-2} - 2(-1)^n\}.$$

It follows that $(L_n)^2$ lies between F_{2n+1} and F_{2n+2}, and is closest to F_{2n+2} provided that $F_{2n-1} + 2(-1)^n > F_{2n-2} - 2(-1)^n$, giving $F_{2n-3} + 4(-1)^n > 0$. This condition is satisfied for $n \geq 4$. Therefore F_{2n+2} is closest.

Also solved by Scott Brown (part (a)), Paul Bruckman, Mario Catalani, Charles Cook, Ovidiu Furdui, Walther Janous, John Jaroma, Harris Kwong, Reiner Martin, Jaroslav Seibert, H.-J. Seiffert and the proposer.

Inequality, Equality Matters

B-943 Proposed by José Luis Diaz & Juan J. Egozcue,
Universitat Politècnica de Catalunya, Terrassa, Spain
(Vol. 40, no. 4, August 2002)

Let n be a positive integer. Prove that

$$\sum_{k=1}^{n} \frac{L_k^2}{F_k} \geq \frac{(L_{n+2} - 3)^2)}{F_{n+2} - 1}.$$

When does equality occur?

Solution by Graham Lord, Princeton, NJ

We shall use the known identities that the sum of the first n Fibonacci numbers is $F_{n+2}-1$, and the sum of the first n Lucas numbers is $L_{n+2} - 3$. (See, for example, pages 52 and 54 of *Fibonacci and Lucas Numbers* by V.E. Hoggatt, 1969.)

Then, by appealing to the Cauchy-Schwarz inequality $(\Sigma ab)^2 \leq (\Sigma a^2) \cdot (\Sigma b^2)$, we have: (all sums are over k from i to n)

$$(L_{n+2} - 3)^2 = \left(\sum L_k\right)^2 = \left(\sum \frac{L_k}{\sqrt{F_k}} \cdot \sqrt{F_{kj}}\right)^2 \leq$$

$$\left(\sum \frac{L_k^2}{F_j}\right)\left(\sum F_k\right) = \left(\sum \frac{L_k^2}{F_k}\right)(F_{n+2} - 1)$$

We will have equality iff the two sets of numbers L_k^2/F_k and F_k are proportional, that is, iff $L_1/F_1 = L_2/F_2 = \cdots = L_n/F_n$. The latter condition is only true iff $n = 1$.

Also solved by Paul Bruckman, Mario Catalani, L.A.G. Dresel, Ovidiu Furdui, Walther Janous, Jaroslav Seibert, H.-J. Seiffert (2 solutions) and the proposer.

Walther Janous proved a more general inequality that may appear as a separate proposal.

A Prime Congruence

B-944 Proposed by Paul S. Bruckman, Berkeley, CA (Vol. 40, no. 4, August 2002)

For all odd primes p, prove that

$$L_p \equiv 1 - \frac{p}{2}\sum_{k=1}^{p-1}\frac{L_k}{k} (\text{mod } p^2)$$

where $\frac{1}{k}$ represents the residue $k^{-1}(\text{mod } p)$.

Solution by H.-J. Seiffert, Berlin, Germany

Let p be an odd prime. From

$$k(p-1)(p-2)\ldots(p-k+1) \equiv (-1)^{k-1}k! \ (\text{mod } p),$$

we obtain the well-known congruence

$$(-1)^k\binom{p}{k} \equiv -\frac{p}{k}(\text{mod } p^2), \ k = 1, 2, \ldots, p-1.$$

Since (see, for example, P. Haukkanen. "On a Binomial Sum for the Fibonacci and Related Numbers." *The Fibonacci Quarterly* **34.4** (1996): 326-31, Corollary 2)

$$L_p = \sum_{k=0}^{p}(-1)^k\binom{p}{k}L_k,$$

modulo p^2, we then have

$$L_p \equiv L_0 + (-1)^p L_p = p\sum_{k=1}^{p-1}\frac{L_k}{k}(\text{mod } p^2).$$

The desired congruence now easily follows by observing that $L_0 = 2$ and that p is odd.

Also solved by L.A.G. Dresel and the proposer.

A Simpler Expression

B-945 Proposed by N. Gauthier, Royal Military College of Canada (Vol. 40, no. 4, August 2002)

For $n \geq 0,\ q > 0,\ s$ integers, show that

$$\sum_{l=0}^{n} \binom{n}{l} F_{q-1}^{l} F_{(q+1)(n-l)+s} = F_{q+1}^{n} F_{2n+s}.$$

Solution I by Paul S. Bruckman, Berkeley, CA

Denote the given sum as $S(n; q, s)$.

Then $S(n; q, s) = 5^{-1/2} \sum_{k=0}^{n} +nC_k (F_{q-1})^k \{\alpha^{s+(q+1)(n-k)} - \beta^{s+(q+1)(n-k)}\} = 5^{-1/2}\{\alpha^s (F_{q-} \alpha^{q+1})^n - \beta^s (F_{q-1} + \beta^{q+1})^n\}$. Now $F_{q-1} + \alpha^{q+1} = F_{q-1} + \alpha F_{q+1} + F_q = F_{q+1}(1+\alpha) = \alpha^2 F_{q+1}$; likewise, $F_{q-1} + \beta^{q+1} = \beta^2 F_{q+1}$. Therefore, $S(n; q, s) = 5^{-1/2} (F_{q+1})^n \{\alpha^{s+2n} - \beta^{s+2n}\} = (F_{q+1})^n F_{2n+s}$.

Solution II by Pentti Haukkanen, University of Tampere, Tampere, Finland

Problem B-945 is a special case of Problem H-121. In fact, according to Problem H-121

$$\sum_{l=0}^{n} \binom{n}{l} \left(\frac{F_{2k}}{F_{m-2k}}\right)^{l} F_{ml+s} = \left(\frac{F_m}{F_{m-2k}}\right)^{n} F_{2nk+s}.$$

Replacing l with $n-1$ we obtain

$$\sum_{l=0}^{n} \binom{n}{l} \left(\frac{F_{2k}}{F_{m-2k}}\right)^{n-l} F_{m(n-l)+s} = \left(\frac{F_m}{F_{m-2k}}\right)^{n} F_{2nk+s}.$$

Writing $m = q+1,\ k = 1$ and multiplying with F_{q-1}^{n} we arrive at the proposed identity

$$\sum_{l=0}^{n} \binom{n}{l} F_{q-1}^{l} F_{(q+1)(n-l)+s} = F_{q+1}^{n} F_{2n+s}.$$

Also solved by Mario Catalani, Kenneth B. Davenport, Ovidiu Furdui, and the proposer.

NOTES

1. We would like to belatedly acknowledge the receipt of a solution to problem B-938 by Jereme Jarome. Also, Kenneth Davenport submitted a late solution to the same problem.
2. Solution I by Pantelimon Stanica to problem B-933 contains a fatal error. In fact, the first inequality in the proof, $F_{n+1}F_{n+1} > F_{n+1}F_n$ should be reversed. We would like to apologize for the oversight.