# A FIBONACCI IDENTITY IN THE SPIRIT OF SIMSON AND GELIN-CESÀRO


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The identities

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n \tag{1}$$

and

$$F_{n-2}F_{n-1}F_{n+1}F_{n+2} - F_n^4 = -1 \tag{2}$$

are very old, dating back to 1753 and 1880, respectively. According to Dickson [1], pages 393 and 401, the first was proved by Robert Simson and the second was stated by E. Gelin and proved by E. Cesàro. Simson's identity and the Gelin-Cesàro identity have been generalized many times. For details and references we refer the reader to [3] and [4].

We began to wonder if there was a pleasing identity involving the difference of *third-order* products, but a search of the literature revealed nothing to match the beauty of the identities above. We offer the following:

$$F_{n+1}F_{n+2}F_{n+6} - F_{n+3}^3 = (-1)^n F_n. \tag{3}$$

One method of proof is to use (1) to substitute for $(-1)^n$, then express each of $F_{n-1}, F_n, F_{n+3}$, and $F_{n+6}$ in terms of $F_{n+1}$ and $F_{n+2}$, and expand both sides. We prefer this method of proof since it carries over nicely to our generalization of (3), which we give next.

Our generalization is stated for the sequence $\{W_n\} = \{W_n(a,b;p,q)\}$ defined by

$$W_n = pW_{n-1} - qW_{n-2}, \quad W_0 = a, \quad W_1 = b$$

where $a, b, p$, and $q$ are taken to be arbitrary complex numbers with $q \neq 0$. Since $q \neq 0$, $\{W_n\}$ is defined for all integers $n$. Put $e = pab - qa^2 - b^2$. Then

$$W_{n+1}W_{n+2}W_{n+6} - W_{n+3}^3 = eq^{n+1}\left(p^3 W_{n+2} - q^2 W_{n+1}\right), \tag{4}$$

which clearly generalizes (3). Generalizations often suffer through a loss of elegance, but this is not the case here, adding testimony to the charm of (3).

To prove (4) we require the identity

$$W_{n+1}W_{n+3} - W_{n+2}^2 = eq^{n+1}, \tag{5}$$

which generalizes (1) and is a variant of (4.3) in [2]. In addition, we require

$$\begin{cases} W_{n+3} = pW_{n+2} - qW_{n+1} \\ W_{n+6} = \left(p^4 - 3p^2q + q^2\right)W_{n+2} - \left(p^3q - 2pq^2\right)W_{n+1} \end{cases} \tag{6}$$

where each identity in (6) is obtained with the use of the recurrence for $\{W_n\}$. Now, using (5) and (6) we express (4) in terms of $p, q, W_{n+1}$, and $W_{n+2}$, and thus verify equality of the left and right sides.

Finally, we remark that Waddill (see (18) in [5]) proved the equivalent of (5) with an elegant use of matrices, which means that our proof of (4) does not rely upon the use of the Binet form ([2]) for $W_n$.

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