ON RATIOS OF FIBONACCI AND LUCAS NUMBERS

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Recently the author has conducted in-service training sessions in mathematics for the elementary school teachers of the Williamstown, Massachusetts public schools. During a session on the lowest common multiple and greatest common divisor of two positive integers, two teachers observed that if the two numbers are in the ratio 2:3, then the sum of the numbers is equal to the difference between their lowest common multiple and their greatest common divisor. It is shown in [2] that this is the only ratio for which this relation holds.

Of course, one gets similar relations for other ratios. For example, if the two numbers are in the ratio 3:5, then twice their sum is equal to the sum of their lowest common multiple and their greatest common divisor. Again it is shown in [2] that this is the only ratio for which this relation holds. This is not always the case since, for example, both ratios 5:7 and 4:11 yield the result that three times the sum of the numbers is equal to the sum of their lowest common multiple and their greatest common divisor.

If one specializes to the Fibonacci and the Lucas sequences, one gets theorems of the type given below, in which families of such relations are exhibited and formulas for finding all ratios satisfying these relations are obtained.

Let $\{F_n\}$ be the sequence of Fibonacci numbers, where $F_1 = 1$, $F_2 = 1$ and $F_{n+2} = F_n + F_{n+1}$ for $n \geq 1$.

Let $\{L_n\}$ be the sequence of Lucas numbers, where $L_1 = 1$, $L_2 = 3$ and $L_{n+2} = L_n + L_{n+1}$ for $n \geq 1$.

The following known results are assumed. (See [1] or [3].)

(i) Neighboring Fibonacci numbers are relatively prime.

(ii) $F_{n+1}^2 = F_nF_{n+2} + (-1)^n$.

(iii) $F_{2n-1} = F_nF_{n+1} - F_{n-1}F_{n-2}$

(iv) Neighboring Lucas numbers are relatively prime.

(v) $F_{2n} = F_nL_n$

(vi) $L_{n+1} = F_n + F_{n+2}$.

For the remainder of the article, let a and b be natural numbers. Denote by $[a, b]$ the lowest common multiple of a and b and by (a, b) the greatest common divisor of a and b.

Theorem 1: (1) If a and b are in the ratio $F_n : F_{n+1}$, then

$$F_{n-1}(a + b) = [a, b] + (-1)^n (a, b) \qquad \text{for } n \geq 2.$$

(2) Let c and d be relatively prime natural numbers with $b = (c/d)a$. If $F_{n-1}(a + b) = [a, b] + (-1)^n (a, b)$ for $n \geq 3$, then the number of solutions for the ratio $c:d$ is one-half the number of divisors of $F_{n-2}F_n$, and among the solutions is the ratio $F_n : F_{n+1}$.

Proof: (1) Suppose $b = (F_n / F_{n+1})a$. Then $a = F_{n+1}k$, $b = F_n k$, $(a,b) = k$ and $[a,b] = F_n F_{n+1} k$, for k a natural number. Then

$$F_{n+1}(a + b) = F_{n-1}(F_n + F_{n+1})k = F_{n-1}F_{n+2}k = (F_{n+1} - F_n)F_{n+2}k$$

$$= F_{n+1}(F_n + F_{n+1})k - F_n F_{n+2} k$$

$$= F_n F_{n+1} k + (F_{n+1}^2 - F_n F_{n+2})k$$

$$= [a, b] + (-1)^n (a, b), \qquad \text{for } n \geq 2.$$

(2) If $b = (c/d)a$, where c and d are relatively prime, then $a = dk$, $b = ck$, $(a, b) = k$ and $[a, b] = cdk$, for k a natural number. Then

$$F_{n-1}(a + b) = [a, b] + (-1)^n (a,b) \qquad \text{for } n \geq 3$$

implies

$$F_{n-1}(c + d) = cd + (-1)^n,$$

for which we wish to find all positive integral solutions. Solving for c, we get

$$c = \frac{F_{n-1}d - (-1)^n}{d - F_{n-1}} = F_{n-1} + \frac{F_{n-1}^2 - (-1)^n}{d - F_{n-1}}$$

so that by (ii),

$$c = F_{n-1} + \frac{F_{n-2}F_n}{d - F_{n-1}}$$

We need only consider the case $d > F_{n-1}$, for if $0 < d < F_{n-1}$, then $c < 0$. The total number of integral solutions for c and d is given by the number of divisors of $F_{n-2}F_n$. But there is an obvious symmetry in these solutions so that if $c = A$, $d = B$ is a solution, then so is $c = B$, $d = A$. Thus the number of distinct solutions for the ratio $c:d$ is one-half the number of divisors of $F_{n-2}F_n$.

Finally, if $d = F_{n+1}$, then

$$c = F_{n-1} + \frac{F_{n-2}F_n}{F_n} = F_{n-1} + F_{n-2} = F_n \quad ,$$

and the ratio $F_n:F_{n+1}$ is among the solutions. This completes the proof.

Example: If $n = 8$, then $F_{n-2} = 8$, $F_{n-1} = 13$, $F_n = 21$, and $F_{n+1} = 34$.

(1) If a and b are in the ratio $21:34$, then

$$13(a + b) = [a,b] + (a,b) \quad .$$

(2) If $b = (c/d)a$, then $13(a + b) = [a,b] + (a,b)$ implies that

$$c = 13 + \frac{168}{d - 13}$$

168 has 16 divisors, so there are 8 distinct solutions. They are: 14:181, 15:97, 16:69, 17:55, 19:41, 20:37, 21:34, and 25:27, among which is the Fibonacci pair 21:34.

The following lemma is needed for the proof of the second theorem.

Lemma: $F_{2n-1} = F_{n+1}L_{n+2} - L_nL_{n+1}$ for $n \geq 2$.

Proof: The proof is by induction. The identity is easily verified for $n = 2$. Assume it is true for $n = k$, so that

$$F_{2k-1} = F_{k+1}L_{k+2} - L_kL_{k+1} \quad .$$

Then

$$\begin{aligned} F_{2k+1} &= F_{2k} + F_{2k-1} = F_kL_k + F_{k+1}L_{k+2} - L_kL_{k+1} \\ &= F_kL_k + (F_{k+2} - F_k)L_{k+2} - L_kL_{k+1} \\ &= F_kL_k + F_{k+2}(L_{k+3} - L_{k+1}) - F_kL_{k+2} - L_kL_{k+1} \\ &= F_{k+2}L_{k+3} - F_{k+2}L_{k+1} - F_kL_{k+1} - (L_{k+2} - L_{k+1})L_{k+1} \\ &= F_{k+2}L_{k+3} - L_{k+1}L_{k+2} + L_{k+1}(L_{k+1} - F_{k+2} - F_k) \quad . \end{aligned}$$

But

$$L_{k+1} - F_{k+2} - F_k = 0$$

by (vi), so that

$$F_{2k+1} = F_{k+2}L_{k+3} - L_{k+1}L_{k+2} \quad ,$$

completing the induction step and the proof.

Theorem 2: (1) If a and b are in the ratio $L_n:L_{n+1}$, then

$$F_{n+1}(a + b) = [a,b] + F_{2n-1}(a,b) \quad \text{for } n \geq 2 \; .$$

(2) If a and b are in the ratio $F_{n-2}:F_{n-1}$, then

$$F_{n+1}(a + b) = [a,b] + F_{2n-1}(a,b) \quad \text{for } n \geq 3 \; .$$

(3) Let c and d be relatively prime natural numbers with

$b = (c/d)a$. If $F_{n+1}(a+b) = [a,b] + F_{2n-1}(a,b)$ for $n \geq 2$, then the possible ratios $c:d$ are determined from the divisors of $(F_{n+1}^2 - F_{2n-1})$. Among these ratios is $L_n:L_{n+1}$. For $n \geq 3$, $F_{n-2}:F_{n-1}$ is also a solution.

Proof: (1) Suppose

$$b = \frac{L_n}{L_{n+1}} a \quad .$$

Then

$$a = L_{n+1}k, \qquad b = L_n k, \qquad (a,b) = k \quad \text{and} \quad [a,b] = L_n L_{n+1} k \quad ,$$

for k a natural number. Then

$$F_{n+1}(a,b) = F_{n+1}(L_n + L_{n+1})k = F_{n+1}L_{n+2}k \quad .$$

Using the lemma, we get

$$F_{n+1}(a+b) = (F_{2n-1} + L_n L_{n+1})k \quad ,$$

so that

$$F_{n+1}(a+b) = [a,b] + F_{2n-1}(a,b) \quad ,$$

as required.

(2) If

$$b = \frac{F_{n-2}}{F_{n-1}} a \quad ,$$

then

$$a = F_{n-1}k \,, \qquad b = F_{n-2}k \,, \qquad (a,b) = k \quad \text{and} \quad [a,b] = F_{n-1}F_{n-2}k$$

for k a natural number. Then, using (iii),

$$F_{n+1}(a + b) = F_{n+1}(F_{n-1} + F_{n-2})k = F_{n+1}F_nk$$
$$= (F_{2n-1} + F_{n-1}F_{n-2})k$$
$$= [a,b] + F_{2n-1}(a,b) \quad ,$$

as required.

(3) If $b = (c/d)a$, where c and d are relatively prime, then, once again

$$a = dk, \qquad b = ck, \qquad (a,b) = k \qquad \text{and} \qquad [a,b] = cdk \quad ,$$

for k a natural number. The relation

$$F_{n+1}(a + b) = [a,b] + F_{2n-1}(a,b)$$

implies

$$F_{n+1}(c + d) = cd + F_{2n-1}$$

Solving this equation for c, we get

$$c = \frac{F_{n+1}d - F_{2n-1}}{d - F_{n+1}} = F_{n+1} + \frac{F_{n+1}^2 - F_{2n-1}}{d - F_{n+1}}$$

We seek positive integral solutions for c and d. The possible ratios $c:d$ are determined from the divisors of $(F_{n+1}^2 - F_{2n-1})$.

Using the lemma, we show that $c = L_n$, $d = L_{n+1}$ is a solution. By symmetry, $c = L_{n+1}$, $d = L_n$ is also a solution. So let $d = L_{n+1}$. Then

$$c = \frac{F_{n+1}L_{n+1} - F_{2n-1}}{L_{n+1} - F_{n+1}} = \frac{F_{n+1}L_{n+1} - F_{n+1}L_{n+2} + L_nL_{n+1}}{L_{n+1} - F_{n+1}}$$

$$= \frac{F_{n+1}(L_{n+1} - L_{n+2}) + L_nL_{n+1}}{L_{n+1} - F_{n+1}} = \frac{-L_nF_{n+1} + L_nL_{n+1}}{L_{n+1} - F_{n+1}} = L_n \quad .$$

The situation here differs from that in the second part of Theorem 1, for not all solutions are obtained by considering the case $d > F_{n+1}$. For example, let $d = F_{n-1}$. Then, using (iii),

$$c = \frac{F_{n+1}F_{n-1} - F_{2n-1}}{F_{n-1} - F_{n+1}} = \frac{F_{n+1}F_{n-1} - F_nF_{n+1} + F_{n-1}F_{n-2}}{F_{n-1} - F_{n+1}}$$

$$= \frac{-F_{n+1}F_{n-2} + F_{n-1}F_{n-2}}{F_{n-1} - F_{n+1}} = F_{n-2} \quad .$$

Thus the ratio $F_{n-2}:F_{n-1}$ is a solution. This completes the proof of the theorem.

Example: If $n = 7$, then $L_n = 29$, $L_{n+1} = 47$, $F_{n-2} = 5$, $F_{n-1} = 8$, $F_{n+1} = 21$ and $F_{2n-1} = 233$.

(1) and (2): If a and b are in the ratio 29:47 or 5:8, then

$$21(a + b) = [a,b] + 233(a,b) \quad .$$

(3): If $b = (c/d)a$, then

$$21(a + b) = [a,b] + 233(a,b)$$

implies that

$$c = 21 + \frac{441 - 233}{d - 21} = 21 + \frac{208}{d - 21}$$

The divisors of 208 are 1, 2, 4, 8, 13, 16, 26, 52, 104 and 208. The solutions are 22:229, 23:125, 25:73, 29:47, 34:37 and 5:8. Among these ratios are $29:47 = L_n:L_{n+1}$ and $5:8 = F_{n-2}:F_{n-1}$.

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