ON A CERTAIN KIND OF FIBONACCI SUMS*

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INTRODUCTION

The sum

$$S(m,n) = \sum_{k=1}^{n} k^m F_k$$

(where F_k is the k^{th} Fibonacci number) has been studied for particular values of m. The cases $m = 0$ and $m = 1$ are well known [1, 2]. The case $m = 3$ was proposed as a problem [3] by Brother U. Alfred of St. Mary's College, California; this problem was later solved [4] by means of translational operator techniques and linear recurrence relations [5]. This method of solution [4] can be generalized for arbitrary positive integral values of m, but it usually will involve the time-consuming, error-inviting procedure of solving $2m + 2$ simultaneous equations in $2m + 2$ variables, which is already a complicated task for $m = 3$.

The method outlined in this paper is much more elementary, and the work required in finding a particular sum is reduced to several simple integrations. The procedure discussed below not only facilitates the computation of these sums, but it is also a useful tool in the solution of other problems, such as the problem of Fibonacci "centroids" proposed by the author [6], certain aspects of Fibonacci convolutions, and the like.

THEORY

Consider the sum

(1) $$\sum_{k=1}^{n} k^m F_k = S(m, n) = F_{n+1}P_2(m, n) + F_nP_1(m, n) + C(m)$$

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where F_k denotes the k^{th} Fibonacci number ($F_0 = 0$, $F_1 = 1$, $F_{k+2} = F_{k+1} + F_k$), $P_1(m,n)$ and $P_2(m,n)$ are polynomials in n of degree m, and $C(m)$ is a constant depending only on the degree m.

Thus we can write

(2a) $$P_1(m,n) = a_m n^m + a_{m-1}n^{m-1} + \cdots + a_1 n + a_0$$

(2b) $$P_2(m,n) = b_m n^m + b_{m-1}n^{m-1} + \cdots + b_1 n + b_0$$

Theorem 1.

$$C(m) = -b_0$$

Proof.

Take $S(m,0) = F_1P_2(m,0) + F_0P_1(m,0) + C(m)$ from (1))

$0 = P_2(m,0) + C(m)$ but $P_2(m,0) = b_0$ from (2b)) .

Inspection of the first few values of m (see Table I) leads us to the following determination of the polynomials (2a) and (2b).

(3a) $$P_1(m,n) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} M_{1,j}n^{m-j}$$

(3b) $$P_2(m,n) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} M_{2,j}n^{m-j}$$

where $\binom{m}{j}$ are the binomial coefficients, and $M_{1,j}$ and $M_{2,j}$ are certain numbers, the law of formation of which is yet to be determined (refer to Table II).

Theorem 2.

(4a) $$P_1(m+1,n) = (m+1)\int_0^n P_1(m,n)dx + a_0'$$

(4b) $$P_2(m+1,n) = (m+1)\int_0^n P_2(m,x)dx + b_0'$$

Table I

LIST OF FIBONACCI SUMS OF THE TYPE

$$S(m,n) = \sum_{k=1}^{n} k^m F_k = F_{n+1}P_2(m,n) + F_nP_1(m,n) + C(m)$$

m = 0 $S(0,n) = F_{n+1}(1) + F_n(1) - 1$

m = 1 $S(1,n) = F_{n+1}(n - 2) + F_n(n - 1) + 2$

m = 2 $S(2,n) = F_{n+1}(n^2 - 4n + 8) + F_n(n^2 - 2n + 5) - 8$

m = 3 $S(3,n) = F_{n+1}(n^3 - 6n^2 + 24n - 50) + F_n(n^3 - 3n^2 + 15n - 31) + 50$

m = 4 $S(4,n) = F_{n+1}(n^4 - 8n^3 + 48n^2 - 200n + 416) + F_n(n^4 - 4n^3 + 30n^2 - 124n + 257) - 416$

m = 5 $S(5,n) = F_{n+1}(n^5 - 10n^4 + 80n^3 - 500n^2 + 2080n - 4322) + F_n(n^5 - 5n^4 + 50n^3 - 310n^2 + 1285n - 2671) + 4322$

m = 6 $S(6,n) = F_{n+1}(n^6 - 12n^5 + 120n^4 - 1000n^3 + 6240n^2 - 25932n + 53888) + F_n(n^6 - 6n^5 + 75n^4 - 620n^3 + 3855n^2 - 16026n + 33305) - 53888$

m = 7 $S(7,n) = F_{n+1}(n^7 - 14n^6 + 168n^5 - 1750n^4 + 14560n^3 - 90762n^2 + 377216n - 783890) + F_n(n^7 - 7n^6 + 105n^5 - 1085n^4 + 8995n^3 - 56091n^2 + 233135n - 484471) + 783890$

m = 8 $S(8,n) = F_{n+1}(n^8 - 16n^7 + 224n^6 - 2800n^5 + 29120n^4 - 242032n^3 + 1508864n^2 - 6271120n + 13031936) + F_n(n^8 - 8n^7 + 140n^6 - 1736n^5 + 17990n^4 - 149576n^3 + 932540n^2 - 3875768n + 8054177) - 13031936$

m = 9 $S(9,n) = F_{n+1}(n^9 - 18n^8 + 288n^7 - 4200n^6 + 52416n^5 - 544572n^4 + 4526592n^3 - 28220040n^2 + 117287424n - 243733442) + F_n(n^9 - 9n^8 + 180n^7 - 2604n^6 + 32382n^5 - 336546n^4 + 2797620n^3 - 17440956n^2 + 72487593n - 150635551) + 243733442$

m = 10 $S(10,n) = F_{n+1}(n^{10} - 20n^9 + 360n^8 - 6000n^7 + 87360n^6 - 1089144n^5 + 11316480n^4 - 94066800n^3 + 586487120n^2 - 2437334420n + 5064892768) + F_n(n^{10} - 10n^9 + 225n^8 - 3720n^7 + 53970n^6 - 673092n^5 + 6994050n^4 - 58136520n^3 + 362437965n^2 - 1506355510n + 3130287705) - 5064892768$

Table II

LIST OF THE $M_{1,j}$ AND $M_{2,j}$ NUMBERS

j	$M_{1,j}$	$M_{2,j}$
0	1	1
1	1	2
2	5	8
3	31	50
4	257	416
5	2671	4322
6	33305	53888
7	484471	783890
8	8054177	13031936
9	150635551	243733442
10	3130287705	5064892768

(5a) $$a_0' = 1 - (m+1)\int_0^1 (P_1(m,x) + P_2(m,x))dx$$

(5b) $$b_0' = 1 - (m+1)\int_0^1 (P_1(m,x) + 2P_2(m,x))dx$$

Proof.

Prove (4a) first. Using (3a) we have

$$(m+1)\int_0^n P_1(m,x)dx = (m+1)\int_0^n \sum_{j=0}^{m} (-1)^j \binom{m}{j} M_{1,j}x^{m-j}dx =$$

$$= (m+1)\sum_{j=0}^{m}(-1)^j M_{1,j}\binom{m}{j}\int_0^n x^{m-j}dx =$$

$$= (m+1)\sum_{j=0}^{m}(-1)^j M_{1,j}\binom{m}{j}\frac{n^{m+1-j}}{m+1-j} = \sum_{j=0}^{m}(-1)^j M_{1,j}\binom{m+1}{j} n^{m+1-j}$$

$$= P_1(m+1,n) - a_0'$$

(a_0' is determined for $j = m + 1$, a value which is missing from the summation sign.) A similar proof establishes (4b).

Now,

$$a_0' = P_1(m + 1, 0) = P_1(m + 1, 1) - (m + 1)\int_0^1 P_1(m, x)dx$$

and

$$b_0' = P_2(m + 1, 0) = P_2(m + 1, 1) - (m + 1)\int_0^1 P_2(m, x)dx$$

and since $S(m + 1, 1) = 1 = P_2(m + 1, 1) + P_1(m + 1, 1) + C(m + 1)$ $(C(m + 1) = -b_0'$ by Theorem 1) then

$$1 = (m + 1)\int_0^1 P_1(m, x)dx + a_0' + (m + 1)\int_0^1 P_2(m, x)dx$$

and the value of a_0' follows. A similar manipulation yields the required value of b_0'.

Corollary 1

$$\frac{dP_1(m + 1, n)}{dn} = (m + 1)\, P_1(m, n);\ \frac{dP_2(m + 1, n)}{dn^r} = m(m + 1)P_2(m, n) .$$

Corollary 2

$$\frac{d^r P_1(m, n)}{dn^r} = m(m - 1)\cdots(m - r + 1)P_1(m - r, n);\ \frac{d^r P_2(m, n)}{dn^r} = m(m - 1)\cdots$$

$$\cdots(m - r + 1)P_2(m - r, n) .$$

Corollary 3

$$P_2(m, 1) = a_0 \quad \text{(refer to (2a, 2b))}.$$

Example 1

Problem. Obtain the sum $\sum_{k=1}^{n} kF_k$.

Solution. We know

$$\sum_{k=1}^{n} F_k = F_{n+1} + F_n - 1 \quad (m = 0) .$$

So the polynomials are $P_1(0,n) = 1$, $P_2(0,n) = 1$. Now, applying Theorem 2,

$$P_1(1,n) = \int_0^n 1dx + a_0' = n + a_0' \quad \text{and} \quad P_2(1,n) = \int_0^n 1dx + b_0' = n + b_0'$$

$$a_0' = 1 - \int_0^1 (1+1)dx = 1 - 2 = -1 \quad \text{and} \quad b_0' = 1 - \int_0^1 (1+2)dx = 1 - 3 = -2$$

Thus, the required sum is equal to $F_{n+1}(n - 2) + F_n(n - 1) + 2$.

Example 2

Problem. Obtain the sum

$$\sum_{k=1}^{n} k^2 F_k .$$

Solution. From Example 1, we know

$$\sum_{k=1}^{n} kF_k = F_{n+1}(n - 2) + F_n(n - 1) + 2$$

So the polynomials are $P_1(1,n) = n - 1$, $P_2(1,n) = n - 2$. Now, applying Theorem 2

$$P_1(2,n) = 2\int_0^n (x - 1)dx + a_0' = n^2 - 2n + a_0' \quad \text{and} \quad P_2(2,n) = 2\int_0^n (x - 2)dx + b_0' = n^2 - 4n + b_0'$$

$$a_0' = 1 - 2\int_0^1 (x - 1 + x - 2)dx = 1 - 2\int_0^1 (2x - 3)dx = 1 - 2(1 - 3) = 1 + 4 = 5$$

$$b_0' = 1 - 2\int_0^1 (x - 1 + 2x - 4)dx = 1 - 2\int_0^1 (3x - 5)dx = 1 - (3 - 10) = 1 + 7 = 8$$

Thus, the required sum is equal to $F_{n+1}(n^2 - 4n + 8) + F_n(n^2 - 2n + 5) - 8.$

Theorem 3.

If u_k are the "generalized" Fibonacci numbers (i. e., numbers obeying the Fibonacci recurrence relation, but with different initial conditions) with the properties $u_{k+2} = u_{k+1} + u_k$, $u_0 = q$, $u_1 = p$, [7], then

$$\sum_{k=1}^{n} k^m u_k = u_{n+1} P_2(m, n) + u_n P_1(m, n) + K(m),$$

where P_2 and P_1 are polynomials defined as above (3a, 3b) and $K(m) = -(pb_0 + qa_0)$.

In Theorem 3 we have stated a simple and useful result. The proof of this theorem is trivial, since $u_k = pF_k + qF_{k-1}$ [7]. Two particular cases are most interesting. The Fibonacci case ($p = 1$, $q = 0$) has been discussed above; the Lucas case ($p = 1$, $q = 2$) is also quite simple (refer to Table III).

At this stage it seems clear that a study of the polynomials $P_1(m, n)$ and $P_2(m, n)$ and of the numbers $M_{1,j}$ and $M_{2,j}$ pose by themselves an interesting problem. The intuitive bounds

$$M_{1,j+1} \geq 2(j + 1)M_{1,j} \qquad M_{2,j+1} \geq 2(j + 1)M_{2,j} \qquad (j \geq 1)$$

hold for all cases shown on Table II and can be proven by total induction using the formulas developed for a_0' and b_0'. A very curious relationship exists between these numbers; this relationship, and the fact that these numbers are members of a whole class of numbers $M_{1,j}$ can be appreciated effectively in Table IV. Horizontal addition of two consecutive $M_{1,j}$ numbers is the basic

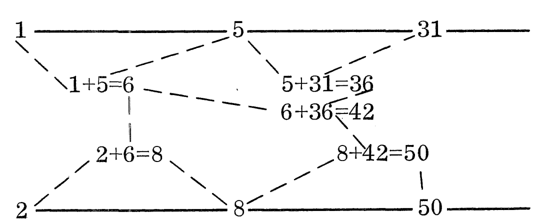
Table III

LIST OF LUCAS SUMS OF THE TYPE

$$T(m,n) = \sum_{k=1}^{n} k^m L_k = L_{n+1}P_2(m,n) + L_nP_1(m,n) + K(m)$$

$m = 0$	$T(0,n) = L_{n+1}(1) + L_n(1) - 3$
$m = 1$	$T(1,n) = L_{n+1}(n - 2) + L_n(n - 1) + 4$
$m = 2$	$T(2,n) = L_{n+1}(n^2 - 4n + 8) + L_n(n^2 - 2n + 5) - 18$
$m = 3$	$T(3,n) = L_{n+1}(n^3 - 6n^2 + 24n - 50) + L_n(n^3 - 3n^2 + 15n - 31) + 112$
$m = 4$	$T(4,n) = L_{n+1}(n^4 - 8n^3 + 48n^2 - 200n + 416) + L_n(n^4 - 4n^3 + 30n^2 - 124n + 257) - 930$
$m = 5$	$T(5,n) = L_{n+1}(n^5 - 10n^4 + 80n^3 - 500n^2 + 2080n - 4322) + L_n(n^5 - 5n^4 + 50n^3 - 310n^2 + 1285n - 2671) + 9664$

principle in the construction of Table IV; the results of successive horizontal additions can be followed with the aid of the broken lines. The following illustration should clarify the process:



These zig-zag relationships imply the second-order linear difference equation

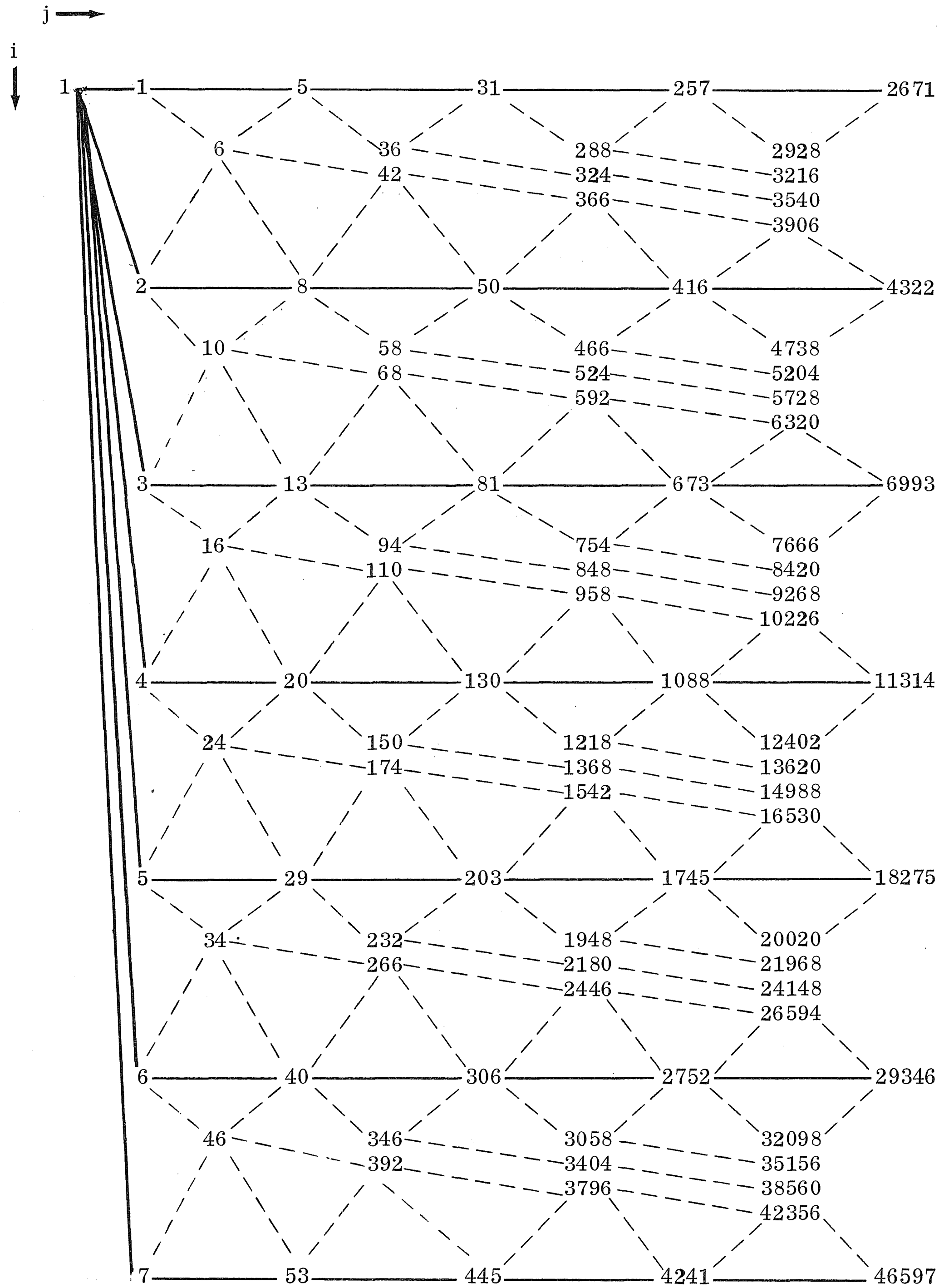
(6a) $$M_{i,j} = M_{i-1,j} + M_{i-2,j} - (i - 3)^j$$

$$(i = 3, 4, 5, \cdots;\ j = 0, 1, 2, \cdots)$$

the solution of which is shown in Eq. (6b).

Table IV

INTERDEPENDENCE CHART FOR THE $M_{i,j}$ NUMBERS



(6b) $$M_{i,j} = F_{i-1}M_{2,j} + F_{i-2}M_{1,j} - \sum_{k=0}^{i-4} (k+1)^j F_{i-3-k}$$

where F_i represents the i^{th} Fibonacci number.

The interdependence of the fundamental set of numbers $M_{1,j}$ and $M_{2,j}$ is noted from the formulas

(6c) $$M_{1,j} = \sum_{h=0}^{j} (-1)^h \binom{j}{h} M_{2,j-h} \quad \text{and} \quad M_{2,j} = \sum_{h=0}^{j} \binom{j}{h} M_{1,j-h}$$

The interdependence of the complete set of numbers $M_{i,j}$ is evidenced with the formula[1]:

(6d) $$M_{i,j} = (i-1)^j + \sum_{h=0}^{j-1} (2^{j-h} - 1) \binom{j}{h} M_{i,h}$$

with $j \geq 0$, $M_{i,0} = 1$, $M_{i,1} = i \geq 1$.

David Zeitlin, in a paper to be published in the Fibonacci Quarterly,[2] has shown that the following relationship holds:

(6e) $$M_{i,j} = \sum_{h=0}^{j} h!\, \mathcal{S}_j^h F_{h+i}$$

where $\mathcal{S}_j^h$ are the Stirling numbers of the second kind.

The polynomials P_1 and P_2 are, similarly, special cases of a more general case of polynomials.

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[1]The author is indebted to Dr. Verner E. Hoggatt, Jr. for pointing out this relationship through personal correspondence.

[2]The author acknowledges the referee for this interesting remark.

(7a) $$P_i(m, n) = \sum_{j=0}^{m} (-1)^j M_{i,j} \binom{m}{j} n^{m-j}$$

which are interrelated in the following ways:

(7b) $$P_{i+h}(m, n) = P_i(m, n - h)$$

(7c) $$P_i(m,n) = P_{i-1}(m,n) + P_{i-2}(m,n) - (n + 3 - i)^m$$

$$(i = 3, 4, 5, \cdots)$$

These properties (7) enable us to obtain the following formula, thus generalize (1):

(8) $$S(m, n - h) = F_{n-h+1}P_{2+h}(m, n) + F_{n-h}P_{1+h}(m, n) + C(m)$$

We have investigated sums of the form

$$F_1 + 2^mF_2 + 3^mF_3 + \cdots + (n - 1)^mF_{n-1} + n^mF_n$$

and it seems quite natural* that we apply our results to the "convolution type" sums of the form

$$n^mF_1 + (n - 1)^mF_2 + (n - 2)^mF_3 + \cdots + 2^mF_{n-1} + F_n \quad .$$

Theorem 4.

(9) $$\sum_{k=1}^{n} (n - k + 1)^mF_k = R(m,n) = M_{3,m}F_{n+1} + M_{2,m}F_n - P_3^*(m, n)$$

*Mathematicians' beloved excuse.

where $M_{3,m}$ and $M_{2,m}$ are particular cases of the $M_{i,j}$ numbers (see Table IV) and $P_3^*(m,n)$ (the "conjugate" of the polynomial $P_3(m,n)$) is defined as follows

$$P_3^{\star}(m,n) = \sum_{j=0}^{m} M_{3,j} \binom{m}{j} n^{m-j} \tag{10}$$

A list of these "convolution-type" sums is provided in Table V.

Table V

LIST OF SUMS OF THE TYPE

$$\sum_{k=1}^{n} (n-k+1)^m F_k = R(m,n) = M_{3,m}F_{n+1} + M_{2,m}F_n - P_3(m,n)$$

$m = 0$	$R(0,n) = F_{n+1} + F_n - 1$
$m = 1$	$R(1,n) = 3F_{n+1} + 2F_n - (n+3)$
$m = 2$	$R(2,n) = 13F_{n+1} + 8F_n - (n^2 + 6n + 13)$
$m = 3$	$R(3,n) = 81F_{n+1} + 50F_n - (n^3 + 9n^2 + 39n + 81)$
$m = 4$	$R(4,n) = 673F_{n+1} + 416F_n - (n^4 + 12n^3 + 78n^2 + 324n + 673)$
$m = 5$	$R(5,n) = 6993F_{n+1} + 4322F_n - (n^5 + 15n^4 + 130n^3 + 810n^2 + 3365n + 6993)$

If $Q(m,n)$ are the Weinshenk polynomials in n of degree m [8], then

$$P_i^*(m,n) = Q(m, n+i-1) \quad \text{and} \quad P_i(m,n) = (-1)^m Q(m, -n+i-1) \tag{11}$$

The above relationships (11) follow from the fact that $P_i^*(m,n) = (-1)^m P_i(m,-n)$. The constant term is then $C(m) = P_1^*(m,1) = Q(m,1)$, and the original sum (1) can be further written as follows:

$$S(m,n) = (-1)^m \{F_{n+1}Q(m,-n+1) + F_nQ(m,-n) - Q(m,1)\} \tag{12}$$

The theoretical interest that these sums arouse is beyond doubt the primary motive for their scrutiny. Weinshenk [8] has applied some of these results to a problem of reflection of light. The problem of centroids [6] can be dealt in a more general manner with the aid of an auxiliary function defined by

$$G(r, s, n) = \frac{\sum_{k=1}^{n} k^r F_k}{\sum_{k=1}^{n} k^s F_k} \tag{13}$$

In particular, $G(1, 0, n) = G_n$ has the following limiting behavior:

$$\lim_{n \to \infty} \frac{G_{n+1}}{G_n} = \lim_{n \to \infty} (G_{n+1} - G_n) = 1 .$$

The problems investigated in this paper are far from being completely solved. Although we could have generalized the subscripts in all our sums [9], we purposely avoided this. However, some questions of importance have not been answered. Some of these questions are:

1. Could the theory of $S(m,n)$ be extended to negative m? (All we need to study is $m = -1$, since the rest of the sums can be obtained with the aid of the algorithms developed in this paper; notice that

$$P_i(-1, n) = \lim_{m \to 0} \frac{\partial^2 P_i(m, n)}{\partial n \partial m} \quad . \quad)$$

2. Could the theory of $S(m, n)$ be extended to rational (and to real) [10] m? If this is possible, what can be said about complex m?

3. What is the possibility of studying sums of the type

$$S(r, s, n) = \sum_{k=1}^{n} k^r F_k^s$$

with the aid of standard techniques?

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