A FIBONACCI FUNCTION

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Eric Halsey [1] has invented an ingenious method for defining the Fibonacci numbers $F(x)$, when x is a rational number. In addition to the restriction that x must be rational, his calculations yield

$$F(4.1) = 3.155, \quad F(3.1) = 2.1, \quad F(1.1) = 1.1 ,$$

so that the Fibonacci identity

$$F(x) = F(x - 1) + F(x - 2)$$

is destroyed.

Fortunately, both of these defects can be remedied, and we can establish a function $F(z)$ which (a) coincides with the usual Fibonacci numbers when Z is an integer, (b) is defined for any <u>complex</u> number z (c) is differentiable everywhere in the complex plane, and (d) is a real number when z is real.

The construction is not difficult. Let λ be the larger of the two roots of

$$y^2 - y - 1 = 0 .$$

Then the Fibonacci formula

$$F(n) = \frac{\lambda^n - (-1)^n \lambda^{-n}}{\sqrt{5}}$$

could be applied directly for n a real number, but would be complex at, for example, $x = 1/2$. By replacing $(-1)^n$ by a real function which takes on the value -1 for n odd and 1 for n even, we can extend $F(n)$ to all real values of n. Such a function is $\cos \pi n$. Hence a Fibonacci <u>function</u> can be written as

$$F(z) = \frac{\lambda^z - (\cos \pi z)\lambda^{-z}}{\sqrt{5}} .$$

An examination of this function shows easily that the stated properties are indeed satisfied.

It is possible to take this one step further. Any solution to the Fibonacci difference equation

$$F(n) = F(n - 1) + F(n - 2)$$

can be similarly treated to yield the equation

$$F(z) = C_1\lambda^z + C_2 (\cos \pi z)\lambda^{-z}$$

where C_1 and C_2 are determined from initial conditions.

In fact, it is possible to generalize even further and produce a similar formula for the solution of the difference equation

$$f(n) = af(n - 1) + bf(n - 2).$$

Such a formula is

$$f(z) = C_1\lambda^z + C_2 b^z (\cos \pi z)\lambda^{-z} ,$$

where λ is a solution to the quadratic equation

$$y^2 - ay - b = 0 .$$

In case these roots are equal, this formula takes the form

$$f(z) = C_1\lambda^z + C_2 z\lambda^z .$$

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1. Eric Halsey, "The Fibonacci Number Fu, where u is not an Integer," The Fibonacci Quarterly, Vol. 3, No. 2, pp. 147-152.

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