ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problem.

H-131 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Consider the left-adjusted Pascal triangle. Denote the left-most column of ones as the zeroth column. If we take sums along the rising diagonals, we get Fibonacci numbers. Multiply each column by its column number and again take sums, C_n, along these same diagonals. Show $C_1 = 0$ and

$$C_{n+1} = \sum_{j=0}^{n} F_{n-j} F_j$$

H-132 Proposed by J.L. Brown, Jr., Ordnance Research Lab., State College, Pa.

Let $F_1 = 1$, $F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$ for $n > 0$. Define the Fibonacci sequence. Show that the Fibonacci sequence is <u>not</u> a basis of order k for any positive integer k; that is, show that <u>not</u> every positive integer can be represented as a sum of k Fibonacci numbers, where repetitions are allowed and k is a fixed positive integer.

H-133 Proposed by V.E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Characterize the sequences

i. $$F_n = u_n + \sum_{j=1}^{n-2} u_j$$

ii. $$F_n = u_n + \sum_{j=1}^{n-2} u_j + \sum_{i=1}^{n-4}\sum_{j=1}^{i} u_j$$

iii. $$F_n = u_n + \sum_{j=1}^{n-2} u_j + \sum_{i=1}^{n-4}\sum_{j=1}^{i} u_j + \sum_{m=1}^{n-6}\sum_{i=1}^{m}\sum_{j=1}^{i} u_j$$

by finding starting values and recurrence relations. Generalize.

H-134 Proposed by L. Carlitz, Duke University

Evaluate the circulants

$$\begin{vmatrix} F_n & F_{n+k} & \cdots & F_{n+(m-1)k} \\ F_{n+(m-1)k} & F_n & \cdots & F_{n+(m-2)k} \\ \cdots & \cdots & \cdots & \cdots \\ F_{n+k} & F_{n+2k} & \cdots & F_n \end{vmatrix}, \quad \begin{vmatrix} L_n & L_{n+k} & \cdots & L_{n+(m-1)k} \\ L_{n+(m-1)k} & L_n & \cdots & L_{n+(m-2)k} \\ \cdots & \cdots & \cdots & \cdots \\ L_{n+k} & L_{n+2k} & \cdots & L_n \end{vmatrix}$$

H-135 Proposed by James E. Desmond, Florida State University, Tallahassee, Fla.

PART I:

Show that

$$j + 1 = \sum_{d=0}^{[j/2]} \binom{j-d}{d} 2^{j-2d}(-1)^d ,$$

where $j \geq 0$ and $[j/2]$ is the greatest integer not exceeding $j/2$.

PART 2:

Show that

$$F_{(j+1)n} = F_n \sum_{d=0}^{[j/2]} \binom{j-d}{d} L_n^{j-2d}(-1)^{(n+1)d}$$

where $j \geq 0$ and $[j/2]$ is the greatest integer not exceeding $j/2$.

SOLUTIONS

RECURSIVE BREEDING

H-89 Proposed by Maxey Brooke, Sweeny, Texas

Fibonacci started out with a pair of rabbits, a male and a female. A female will begin bearing after two months and will bear monthly thereafter. The first litter a female bears is twin males, thereafter she alternately bears female and male.

Find a recurrence relation for the number of males and females born at the end of the n^{th} month and the total rabbit population at that time.

Solution by F. D. Parker, St. Lawrence University

The number of females at the end of n months, $F(n)$, is equal to the number of females at the end of the previous plus the number of females who are at least three months old. Thus we have

$$F(n) = F(n-1) + F(n-3) .$$

The number of males at the end of n months, $M(n)$, will be the sum of the males at the end of the previous month, the number of females at least three months old, and twice the number of females who are exactly two months old. Thus

$$M(n) = M(n-1) + F(n-3) + 2(F(n) - F(n-2)) .$$

The total rabbit population is the same as it would be if each pair of offspring were of mixed sex, that is,

$$M(n) + F(n) = 2f(n) ,$$

where $f(n)$ is the n^{th} Fibonacci number.

DIVIDED WE FALL!

H-92 Proposed by Brother Alfred Brousseau, St. Mary's College, California

Prove or disprove: Apart from F_1, F_2, F_3, F_4, no Fibonacci number, F_i $(i > 0)$ is a divisor of a Lucas number.

Solution by L. Carlitz, Duke University

Put

$$L_n = \alpha^n + \beta^n, \quad F_n = (\alpha^n - \beta^n)/(\alpha - \beta),$$

where

$$\alpha = \tfrac{1}{2}(1 + \sqrt{5}), \quad \beta = \tfrac{1}{2}(1 - \sqrt{5}) .$$

Also put $n = mk + r$, $0 \le r < k$. Since

$$\alpha^n + \beta^n = \alpha^r(\alpha^{mk} - \beta^{mk}) + \beta^{mk}(\alpha^r + \beta^r) ,$$

it follows from $F_k | L_n$ that $F_k | \beta^{mk} L_r$. Since β is a unit of $Q(\sqrt{5})$ it follows that $F_k | L_r$. Now from $L_r = F_{r-1} + F_{r+1}$ we get $L_r < F_{r+2}$ for $r > 2$. Hence we need only consider $F_{r+1} | L_r$. However this implies $F_{r+1} | F_{r-1}$ which is impossible for $r \ge 2$. Therefore $F_k \mid L_n$ is impossible for $k > 4$.

Also solved by James Desmond.

OOPS!!

H-93 Proposed by Douglas Lind, Univ. of Virginia, Charlottesville, Virginia. (corrected).

Show that

$$F_n = \overline{\prod_{k=1}^{n-1}} (3 + 2 \cos 2k\pi/n)$$

$$L_n = \prod_{k=0}^{\overline{n-2}} (3 + \cos (2k + 1)\pi/n)$$

where $\bar{n}$ is the greatest integer contained in $n/2$.

Solution by M.N.S. Swamy, Nova Scotia Technical College, Halifax, Canada.

We know from Problem H-64 (FQ. Vol. 3, April 1965, p. 116) that,

$$F_n = \prod_{j=1}^{n-1} \left(1 - 2i \cos \frac{j\pi}{n} \right),$$

where $i = \sqrt{-1}$.

If n is odd,

$$\begin{aligned} F_{2n+1} &= \prod_{j=1}^{2n} \left(1 - 2i \cos \frac{j\pi}{2n + 1} \right) \\ &= \prod_{1}^{n} \left(1 - 2i \cos \frac{j\pi}{2n + 1} \right) \prod_{n+1}^{2n} \left(1 - 2i \cos \frac{j\pi}{2n + 1} \right) \\ &= \prod_{j=1}^{n} \left(1 - 2i \cos \frac{j\pi}{2n + 1} \right) \prod_{k=n+1}^{2n} \left[1 + 2i \cos \pi\left(1 - \frac{k}{2n + 1} \right) \right] \end{aligned}$$

Letting $j = (2n + 1 - k)$ in the second product we get

$$\begin{aligned} F_{2n+1} &= \prod_{1}^{n} \left(1 - 2i \cos \frac{j\pi}{2n + 1} \right) \prod_{1}^{n} \left(1 + 2i \cos \frac{j\pi}{2n + 1} \right) \\ &= \prod_{1}^{n} \left(1 + 4 \cos^2 \frac{\pi j}{2n + 1} \right) = \prod_{1}^{n} \left(3 + 2 \cos \frac{2j\pi}{2n + 1} \right) \quad \cdots \text{(A)} \end{aligned}$$

Similarly when n is even,

$$F_{2n} = \prod_{j=1}^{2n-1}\left(1 - 2i\cos\frac{j}{2n}\right)$$

$$= \prod_{1}^{n-1}\left(1 - 2i\cos\frac{j\pi}{2n}\right)\prod_{1}^{n-1}\left(1 + 2i\cos\frac{j\pi}{2n}\right)\cdot\left(1 + 2i\cos\frac{\pi}{2}\right)$$

$$= \prod_{1}^{n-1}\left(1 + 4\cos^2\frac{j\pi}{2n}\right)$$

$$= \prod_{1}^{n-1}\left(3 + 2\cos\frac{2j\pi}{2n}\right) \qquad \cdots \text{(B)}$$

From (A) and (B) we see that

$$F_n = \prod_{k=1}^{\overline{n-1}}\left(3 + 2\cos\frac{2k\pi}{n}\right) \qquad \cdots \text{(C)}$$

Hence,

$$F_{2n} = \prod_{k=1}^{\overline{2n-1}}\left(3 + 2\cos\frac{k\pi}{n}\right)$$

$$= \prod_{i=2,4,\cdots 2\overline{(n-1)}}\left(3 + 2\cos\frac{i\pi}{n}\right)\prod_{j=1,3,\cdots,2,\overline{(n-2)}+1}\left(3 + \cos\frac{j\pi}{n}\right)$$

Letting $i = 2k$ and $j = (2k + 1)$ we have

$$F_{2n} = \prod_{k=1}^{\overline{n-1}} \left(3 + 2\cos\frac{2k\pi}{n}\right) \prod_{k=0}^{\overline{n-2}} \left(3 + 2\cos\frac{(2k+1)\pi}{n}\right)$$

$$= F_n \prod_{k=0}^{\overline{n-2}} \left\{3 + 2\cos\frac{(2k+1)\pi}{n}\right\}$$

Since $F_{2n} = F_n L_n$, we have

$$L_n = \prod_{k=0}^{\overline{n-2}} \left[3 + 2\cos\frac{(2k+1)\pi}{n}\right] \qquad \cdots \text{(D)}$$

Also solved by L. Carlitz.

ANOTHER IDENTITY

H-95 Proposed by J. A. H. Hunter, Toronto, Canada.

Show

$$F^3_{n+k} + (-1)^k F^3_{n-k} = L_k F^2_k F_{3n} + (-1)^k F^3_n .$$

Solution by M.N.S. Swamy, Nova Scotia Technical College, Halifax, Canada.

$$\begin{aligned} F_{n-k} &= F_n F_{-(k+1)} + F_{-k} F_{n+1} \\ &= (-1)^k F_n F_{k+1} + (-1)^{k-1} F_k F_{n+1} , \end{aligned}$$

since

$$F_{-n} = (-1)^{n-1} F_n .$$

Hence,

$$(-1)^k F_{n-k} = F_n F_{k+1} - F_k F_{n+1} .$$

Also,

$$F_{n+k} = F_nF_{k-1} + F_kF_{n+1} .$$

Hence we have,

$$F^3_{n+k} + (-1)^{3k} F^3_{n-k} = (F_nF_{k-1} + F_kF_{n+1})^3 + (F_nF_{k+1} - F_kF_{n+1})^3$$

Or,

$$\begin{aligned} I = F^3_{n+k} + (-1)^kF^3_{n-k} &= F^3_n(F^3_{k+1} + F^3_{k-1}) \\ &\quad + 3F_nF_kF_{n+1}F_{k-1}(F_nF_{k-1} + F_kF_{n+1}) \\ &\quad - 3F_nF_kF_{n+1}F_{k+1}(F_nF_{k+1} - F_kF_{n+1}) \\ &= F^3_n(F_{k+1} + F_{k-1})(F^2_{k+1} - F^2_{k-1} - F_{k+1}F_{k-1}) \\ &\quad - 3F^2_nF_kF_{n+1}(F^2_{k+1} - F^2_{k-1}) + 3F_nF^2_kF^2_{n+1}(F_{k+1} + F_{k-1}) \\ &= F^3_n L_k (F_{k+1} - F_{k-1})^2 + F_{k+1} F_{k-1} \\ &\quad - 3F^2_nF_kF_{n+1}(F_{k+1} + F_{k-1})(F_{k+1} - F_{k-1}) \\ &\quad + 3F_nF^2_kF^2_{n+1}L_k \\ &= L_kF^3_n(F^2_k + F_{k+1}F_{k-1}) - 3F^2_nF^2_kF_{n+1}L_k + 3F_nF^2_kF^2_{n+1}L_k \end{aligned}$$

Using the identity,

$$F^2_k - F_{k+1}F_{k-1} = (-1)^k$$

we obtain

$$\begin{aligned} I &= L_kF^3_n\left(2F^2_k + (-1)^k\right) + 3L_kF^2_kF_nF_{n+1}(F_{n+1} - F_n) \\ &= L_kF^2_k(2F^3_n + 3F_nF_{n+1}F_{n-1}) + (-1)^kF^3_nL_k \end{aligned} \tag{1}$$

Now,

$$\begin{aligned}
F_{3n} &= F_n F_{2n-1} + F_{2n} F_{n+1} \\
&= F_n(F_{n-1}^2 + F_n^2) + (F_n F_{n-1} + F_n F_{n+1}) F_{n+1} \\
&= F_n^3 + F_n F_{n+1}^2 + F_{n-1} F_n (F_{n+1} + F_{n-1}) \\
&= F_n^3 + F_n(F_n^2 + 2F_n F_{n-1} + F_{n-1}^2) + F_{n-1} F_n (F_{n+1} + F_{n-1}) \\
&= 2F_n^3 + 2F_{n-1} F_n (F_{n-1} + F_n) + F_{n-1} F_n F_{n+1} \\
&= 2F_n^3 + 3F_{n-1} F_n F_{n+1}
\end{aligned}$$

Substituting this in (1) we get

$$I = L_k F_k^2 F_{3n} + (-1)^k F_n^3 L_k$$

Therefore,

$$F_{n+k}^3 + (-1)^k F_{n-k}^3 = L_k \left[F_k^2 F_{3n} + (-1)^k F_n^3 \right]$$

Also solved by Charles R. Wall.

LATE ACKNOWLEDGEMENTS

Clyde Bridger: H-79, H-80.

C. B. A. Peck: H-32, H-44, H-45, H-67.

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(Continued from p. 138.)

All known Fibonacci equations using F_n are theoretically generalizable to equations using F_x. For some examples, see [2]. See [3] also.

REFERENCES

1. V. E. Hoggatt, Jr., and Douglas Lind, "Power Identities for Second-Order Recurrent sequences," Fibonacci Quarterly, Vol. 4, No. 3, Oct. 1966.
2. Allan Scott, "Fibonacci Continuums," unpublished.
3. F. D. Parker, "Fibonacci Functions," Fibonacci Quarterly, Vol. 6, No. 1, pp. 1-2.

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