A THEOREM ON POWER SUMS

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Allison [1, p. 272] showed that the identity

$$\left\{\sum_{x=1}^{n} x^r\right\}^p = \left\{\sum_{x=1}^{n} x^s\right\}^q \qquad (n = 1, 2, 3, \cdots) \tag{1}$$

holds if and only if $r = 1$, $p = 2$, $s = 3$, and $q = 1$. In this paper we consider the more general problem of finding polynomials

$$f(x) = \sum_{i=0}^{r} a_i x^i \qquad \text{and} \qquad g(x) = \sum_{i=0}^{s} b_i x^i$$

over the real field which satisfy

$$\{f(1) + \cdots + f(n)\}^p = \{g(1) + \cdots + g(n)\}^q \quad (n = 1, 2, 3, \cdots), \tag{2}$$

where r, p, s and q are positive integers.

First we note that

$$\sum_{x=1}^{n} f(x) = \sum_{i=0}^{r} a_i S_i ,$$

where

$$S_k = \sum_{x=1}^{n} x^k, \qquad k = 0, 1, 2, \cdots .$$

Thus the left member of (2) becomes

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$$\left\{a_r \frac{n^{r+1}}{r+1} + \cdots\right\}^p ,$$

since S_r is a polynomial in n having degree $r + 1$ and leading coefficient

$$\frac{1}{r+1} .$$

Similarly the right member of (2) becomes

$$\left\{b_s \frac{n^{s+1}}{s+1} + \cdots\right\}^q ,$$

so (2) can be written

$$\left\{a_r \frac{n^{r+1}}{r+1} + \cdots\right\}^p = \left\{b_s \frac{n^{s+1}}{s+1} + \cdots\right\}^q . \tag{3}$$

For (3) to hold we must have

$$(r + 1)p = (s + 1)q \tag{4}$$

and

$$\left(\frac{a_r}{r+1}\right)^p = \left(\frac{b_s}{s+1}\right)^q . \tag{5}$$

Case 1. Suppose $p = q$. From (2) we find $f(n) = g(n)$, $n = 1, 2, 3, \cdots$, so $f(x) = g(x)$.

Case 2. Suppose $p \neq q$. We may assume without loss of generality that $p > q$ and $(p, q) = 1$. We will also assume that $a_r = b_s = 1$. Following Allison [op. cit.] we see that for (3) to hold we must have $r = 1$, $p = 2$, $s = 3$, and $q = 1$. Specifically,

$$(S_1 + a_0S_0)^2 = S_3 + b_2S_2 + b_1S_1 + b_0S_0 . \tag{6}$$

Using well-known formulas for S_k, $k = 0, 1, 2, 3$, we write (6) as

$$\left\{\frac{n(n+1)}{2} + a_0 n\right\}^2 = \left\{\frac{n(n+1)}{2}\right\}^2 + b_2\left\{\frac{n(n+1)(2n+1)}{6}\right\} + b_1 \frac{n(n+1)}{2} + b_0 n. \tag{7}$$

Rewriting (7) in powers of n, we find

$$\begin{aligned} \frac{n^4}{4} + \left(\frac{1}{2} + a_0\right)n^3 + \left(\frac{1}{2} + a_1\right)^2 n^2 &= \frac{n^4}{4} + \left(\frac{1}{2} + \frac{b_2}{3}\right)n^3 \\ &+ \left(\frac{1}{4} + \frac{b_2}{2} + \frac{b_1}{2}\right)n^2 + \left(\frac{b_2}{6} + \frac{b_1}{6} + b_0\right)n . \end{aligned} \tag{8}$$

Equating coefficients in (8) yields

$$\begin{aligned} a_0 &= \frac{b_2}{3} \\ \left(\frac{1}{2} + a_0\right)^2 &= \frac{1}{4} + \frac{b_2}{2} + \frac{b_1}{2} \\ 0 &= \frac{b_2}{6} + \frac{b_1}{2} + b_0 . \end{aligned} \tag{9}$$

Let a_0 be arbitrary and regard (9) as the linear system

$$\sum_{j=0}^{2} a_{ij} b_j = c_i \qquad (i = 0, 1, 2). \tag{10}$$

Since the determinant $|a_{ij}| \neq 0$, we can solve for b_0, b_1, b_2 in terms of a_0. Easy calculations show

$$b_0 = -a^2 , \quad b_1 = 2a^2 - a, \quad b_2 = 3a , \tag{11}$$

where a_0 is replaced by a for simplicity. Thus

$$f(x) = x + a, \quad g(x) = x^3 + 3ax^2 + (2a^2 - a)x - a^2 \tag{12}$$

When $a = 0$, (12) yields the result of Allison.

If we do not require $a_r = b_s = 1$, it is interesting to note that for arbitrary p, q one can always find non-monic polynomials $f(x)$, $g(x)$ to satisfy (2). Specifically $f(x)$ and $g(x)$ are chosen to satisfy

$$\sum_{x=1}^{n} g(x) = n^q, \quad \sum_{x=1}^{n} g(x) = n^p . \tag{13}$$

If (13) holds, obviously (2) does.

In general the construction of a function $f_t(x)$ satisfying

$$\Sigma f_t(x) = n^t \qquad (t = 1, 2, 3, \cdots) \tag{14}$$

is recursive. First note that $f_1(x) = 1$. We find $f_{t+1}(x)$ as follows. Recall that

$$\sum_{x=1}^{n} x^t = \frac{n^{t+1}}{t+1} + s_t n^t + \cdots + s_1 n .$$

Thus

$$(t+1)\sum_{x=1}^{n}\left\{x^t - s_t f_t(x) - \cdots - s_1 f_1(x)\right\} = n^{t+1} , \tag{15}$$

so

$$f_{t+1}(x) = (t+1)\left\{x^t - \sum_{k=1}^{t} s_k f_k(x)\right\} . \tag{16}$$

We summarize these results in the following.

Theorem. The solutions of (2) are as follows. If $p = q$, $f(x)$ is arbitrary and $g(x) = f(x)$. If $p \neq q$, the only monic solutions occur when $p = 2$ and $q = 1$, in which case $f(x)$ and $g(x)$ are defined by (12), where a is an arbitrary real constant. Non-monic solutions for that case can be found using (13).

As an example of these results suppose that $p = 3$ and $q = 4$. By (14) and (17) we have

$$\left\{\sum_{x=1}^{n}(4x^3 - 6x^2 + 4x - 1)\right\}^3 = \left\{\sum_{x=1}^{n}(3x^2 - 3x + 1)\right\}^4, \quad (n = 1, 2, 3, \cdots).$$

REFERENCE

1. Allison, "A Note on Sums of Powers of Integers," American Mathematical Monthly, Vol. 68, 1961, p. 272.

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A NUMBER PROBLEM

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There are infinite many numbers with the property: if units digit of a positive integer, M, is 6 and this is taken from its place and put on the left of the remaining digits of M, then a new integer, N, will be formed, such that $N = 6M$. The smallest M for which this is possible is a number with 58 digits (1016949 ⋯ 677966).

Solution: Using formula

$$\frac{6x}{1 - 4x - x^2} = 3\sum_{n=0}^{\infty} F_{3n}\, x^n ,$$

with $x = 0,1$ we have $1,01016949 \cdots 677966$, where the period number (behind the first zero) is M.*

*1016949152542372881355932203389830508474576271186440677966.

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