ON A CERTAIN INTEGER ASSOCIATED WITH A GENERALIZED FIBONACCI SEQUENCE

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1. INTRODUCTION

A generalized Fibonacci sequence may be defined by specifying two relatively prime integers and applying the formula

$$y_n = py_{n-1} + y_{n-2}, \tag{1}$$

where p is a fixed positive integer ($p = 1$ gives a Fibonacci sequence).

If y_0 is the smallest non-negative term determined by (1), then $y_1 \geq (p + 1)y_0$ with strict inequality for $y_0 > 1$ except in the case $y_0 = y_1 = 1$. In order to avoid trivial exceptions to various statements below, we assume with no real loss of generality that $y_1 > y_0 > 0$ in all that follows.

It has been shown in [1] that the Fibonacci sequences can be ordered using the quantity $y_1^2 - y_0y_1 - y_0^2$. Similarly, the generalized Fibonacci sequences defined in (1) may be ordered using the quantity D defined by

$$D = y_1^2 - py_0y_1 - y_0^2 .$$

It may be of interest to determine for given p the possible values of D and how many generalized Fibonacci sequences can be associated with a given value of D.

We solve completely the cases $p = 1, 2$ which, as will be seen, are essentially simpler than the cases $p \geq 3$. Our proofs make use of the classical theory of binary quadratic forms of positive discriminant

$$d = p^2 + r .$$

A good treatment of this subject is found in [2], which we refer to frequently as a source of the proofs of well-known results.

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Let S_p be the set of positive integers D such that the congruence

$$n^2 \equiv d \mod 4D$$

has solutions for n. We prove the following:

Theorem 1. For $p = 1, 2$, S_p is the set of possible values of the integer $D = y_1^2 - py_0y_1 - y_0^2$ associated with the generalized Fibonacci sequence defined by (1).

Theorem 2. For $p = 1, 2$, let r be the number of distinct odd primes dividing $4D/(d, 4D)$. Then except for the trivial case $p = D = 2$ there are 2^{r+1-p} distinct pairs y_0, y_1 such that $D = y_1^2 - py_0y_1 - y_0^2$ and y_0, y_1 generate a generalized Fibonacci sequence defined by (1), i.e., there are 2^{r+1-p} distinct sequences associated with the given value of D.

The case $p = 1$ of Theorem 1 has been previously proved in [3].

2. REMARKS FOR THE CASE OF GENERAL p

Our problem is to determine all positive integers D which are properly represented (i.e., are represented with x and y relatively prime) by the form

$$Q = x^2 - pxy - y^2$$

with the restriction that

$$x \geq (p + 1)y \geq 0 \tag{2}$$

We denote the quadratic form $ax^2 + bxy + cy^2$ by (a, b, c). We say the ordered pair $(x, y) = (\alpha, \gamma)$ is a proper representation of m by (a, b, c) if α and γ are relatively prime and $a\alpha^2 + b\alpha\gamma + c\gamma^2 = m$.

Lemma 1. Let (α, γ) be a proper representation of the positive integer D by the integral form (a, b, c) of discriminant d. Then there exist unique integers β, δ, n satisfying

$$\begin{aligned} &\alpha\delta - \beta\gamma = 1 \\ (3)\qquad &0 \leq n < 2D \end{aligned}$$

(4) $$n^2 \equiv d \mod 4D$$

and such that the transformation

(5) $$x = \alpha x' + \beta y'$$
$$y = \gamma x' + \delta y'$$

replaces (a, b, c) by the equivalent form (D, n, k) in which k is determined by

$$n^2 - 4Dk = d$$

Proof. This is a classical result ([2, p. 74, Th. 58]).

Corollary. Q properly represents a positive integer D only if D belongs to the set S_p.

Following [2, p, 74] we call a root n of (4) which satisfies (3) a minimum root. Since n is a root of (4) if and only if n + 2D is also a root, the number of minimum roots is half the total number of roots. By Lemma 1, a proper representation of D by a form (a, b, c) is associated with a unique minimum root of (4).

Lemma 2. Every automorph (5) of the integral form (a, b, c) of discriminant d, where a, b, c have no common divisor 1, has

(6) $$\alpha = \tfrac{1}{2}(u - bv) \qquad \beta = -cv \qquad \gamma = av \qquad \delta = \tfrac{1}{2}(u + bv)\ ,$$

where u and v are integral solutions of

(7) $$u^2 - dv^2 = 4\ .$$

Conversely, if u and v are integral solutions of (7), the numbers (6) are integers and define an automorph.

Proof. This is a classical result ([2, p. 112, Th. 87]).

Lemma 3. For given D in S_p, there is associated with a given minimum root n of (4) at most one proper representation of D by (1, -p, -1), which satisfies (2).

Proof. Let (α, γ) be a proper representation of D by $(1, -p, -1)$ satisfying (2) and associated with the minimum root n of (4). For the given D and n, it is clear that any proper representation (α', γ') of D by $(1, -p, -1)$ is the first column of a matrix

$$A\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{bmatrix},$$

where A is the matrix of some automorph of $(1, -p, -1)$. Thus it is enough to show that (α', γ') does not satisfy (2) unless A is the identity matrix.

Since the smallest positive solution of the equation (7) is obviously $(u, v) = (p^2 + 2, p)$, it follows from Lemma 2 that every automorph of $(1, -p, -1)$ is of the form

$$A = \begin{bmatrix} p^2 + 1 & p \\ p & 1 \end{bmatrix}^m \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}^j \quad \begin{matrix} j = 1 \text{ or } 2 \\ m = 0, \pm 1, \pm 2, \cdots \end{matrix}$$

We need only consider non-negative m, because for negative m (α', γ') clearly has components of opposite sign. Obviously (α', γ') does not satisfy (2) for $j = 1$ and any $m \geq 0$. For $j = 2$, $m = 0$, $(\alpha', \gamma') = (\alpha, \gamma)$ satisfies (2) by hypothesis; but this is false for $j = 2$, $m = 1$ because

$$(p + 1)(p\alpha + \gamma) \geq (p^2 + 1)\alpha + p\gamma.$$

Then by induction (α', γ') does not satisfy (2) for $j = 2$ and any $m \geq 1$. This proves the lemma.

3. CASE p = 1 OF THEOREM 1

Lemma 4. S_1 is made up of

1. The integers 1 and 5
2. all primes $\equiv 1$ or 9 mod 10
3. all products of the above integers $\not\equiv 0$ mod 25.

Proof. By definition, S_1 is the set of positive integers D such that the congruence

$$n^2 \equiv 5 \mod 4D \tag{8}$$

has solutions for n. Thus we must have $D \not\equiv 0 \mod 25$ and D odd, since

$$\left(\frac{5}{8}\right) = -1 .$$

So it is enough to show that (8) is soluble for odd prime D if and only if D = 5, or $D \equiv 1$ or 9 mod 10.

By the definition of the Legendre symbol, (8) is soluble for odd prime D if and only if

$$\left(\frac{5}{D}\right) = 1 .$$

But then by quadratic reciprocity and the fact that D is odd

$$\left(\frac{5}{D}\right) = \left(\frac{D}{5}\right) = \begin{cases} 1 & \text{if } D \equiv 1 \text{ or } 4 \mod 5 \\ -1 & \text{if } D \equiv 2 \text{ or } 3 \mod 5 \end{cases}$$

which implies the desired result.

Lemma 5. If D belongs to S_1, then (1, -1, -1) properly represents D. Further, associated with each minimum root of (8) there is at least one proper representation satisfying (2) with p = 1.

Proof. We consider each of the minimum roots of (8). Let (α, γ) be a proper representation of D by (1, -1, -1) associated with a given minimum root n.

We may suppose $\alpha > 0$, $\gamma > 0$. For if $\alpha < 0$, $\gamma < 0$, we consider $(-\alpha, -\gamma)$. If one and only one of α, γ is negative we may suppose it is α. Then we apply the automorph

$$\begin{aligned} x' &= 2x + y \\ y' &= x + y \end{aligned} \tag{9}$$

of (1, -1, -1) successively to (α, γ), getting the sequence

$$(\alpha, \gamma),\ (2\alpha + \gamma,\ \alpha + \gamma),\ \cdots,\ (f_{2m+1}\alpha - f_{2m}\gamma,\ f_{2m}\alpha + f_{2m-1}\gamma),\ \cdots$$

where f_i is the i^{th} member of the Fibonacci sequence $1, 1, 2, 3, 5, \cdots$, If for some m we have

$$f_{2m} |\alpha| > f_{2m-1}\gamma , \tag{10}$$

then

$$(-f_{2m+1}\alpha - f_{2m}\gamma , -f_{2m}\alpha - f_{2m-1}\gamma)$$

is a proper representation with both members positive, as desired. But (10) must be true for some m because $\gamma = k|\alpha|$ for some rational $k > 0$ and

$$\alpha^2 - \alpha\gamma - \gamma^2 > 0$$

implies

$$k < (1 + \sqrt{5})/2 ;$$

whereas from the continued fraction expansion of $(1 + \sqrt{5})/2$ we have

$$1 < \frac{3}{2} < \frac{8}{5} < \cdots < \frac{f_{2m}}{f_{2m-1}} < \cdots < \frac{1 + \sqrt{5}}{2}$$

and

$$\lim_{m \to \infty} \frac{f_{2m}}{f_{2m-1}} = \frac{1 + \sqrt{5}}{2} .$$

Given a proper representation (α, γ) with both members positive, we apply the inverse of the transformation (9) successively, getting the sequence

$$(\alpha, \gamma), \quad (\alpha - \gamma, -\alpha + 2\gamma) , \cdots ,$$

$$(f_{2m-1}\alpha - f_{2m}\gamma, -f_{2m}\alpha + f_{2m+1}\gamma), \cdots$$

Since the successive first members make up a decreasing sequence of positive integers so long as the corresponding second members are positive, we must reach an m such that

$$f_{2m+1}\gamma > f_{2m}\alpha \quad \text{and} \quad f_{2m+3}\gamma < f_{2m+2}\alpha .$$

Then

$$(f_{2m+1}\alpha - f_{2m}\gamma ,\ -f_{2m}\alpha + f_{2m+1}\gamma)$$

is a proper representation satisfying (2) with $p = 1$.

All transformations used above of course have determinant 1, so that the minimum root n associated with the originally given proper representation is not changed.

4. CASE $p = 2$ OF THEOREM 1

Lemma 6. S_2 is made up of

1. the integers 1 and 2
2. all primes $\equiv 1$ or 7 mod 8
3. all products of the above integers $\not\equiv 0$ mod 4.

Proof. By definition, S_2 is the set of positive integers D such that the congruence

$$n^2 \equiv 8 \mod 4D \tag{11}$$

has solutions for n. Thus we must have $D \not\equiv 0$ mod 4. Then the result follows from the fact that for odd prime D

$$\left(\frac{2}{D}\right) = \begin{cases} 1 & \text{if } D \equiv 1 \text{ or } 7 \bmod 8 \\ -1 & \text{if } D \equiv 3 \text{ or } 5 \bmod 8 \end{cases}$$

Lemma 7. If D belongs to S_2, then $(1, -2, -1)$ properly represents D. Further, associated with exactly half of the total number of minimum roots of (11) there is at least one proper representation satisfying (2) with $p = 2$.

Proof. We consider each of the minimum roots of (11). Let (α, γ) be a proper representation of D by $(1, -2, -1)$ associated with a given minimum root n.

We argue as in Lemma 5 that we may suppose $\alpha < 0$, $\gamma < 0$. For if $\alpha < 0$, $\gamma < 0$ we consider $(-\alpha, -\gamma)$. If one and only one of α, γ is negative, we may suppose it is α. Then we apply the automorph

$$\begin{aligned} x' &= 5x + 2y \\ y' &= 2x + y \end{aligned} \tag{12}$$

of $(1, -2, -1)$ successively to (α, γ), getting the sequence

$$\begin{gathered} (\alpha, \gamma),\ (5\alpha + 2\gamma,\ 2\alpha + \gamma), \cdots, \\ (g_{2m+1}\alpha + g_{2m}\gamma,\ g_{2m}\alpha + g_{2m-1}\gamma), \cdots \end{gathered}$$

where g_i is the i^{th} member of the generalized Fibonacci sequence 1, 2, 5, 12, 29, $\cdots$. If for some m we have

$$g_{2m}|\alpha| > g_{2m-1}\gamma , \tag{13}$$

then

$$(-g_{2m+1}\alpha - g_{2m}\gamma,\ -g_{2m}\alpha - g_{2m-1}\gamma)$$

is a proper representation with both members positive. But as in the proof of Lemma 5 a consideration of the continued fraction for $1 + \sqrt{2}$ shows that (13) must be true for some m.

Given a proper representation (α, γ) with both members positive, we apply the inverse of the transformation (12) successively, getting the sequence

$$(\alpha, \gamma),\ (\alpha - 2\gamma,\ -2\alpha + 5\gamma), \cdots, (g_{2m-1}\alpha - g_{2m}\gamma,\ -g_{2m}\alpha + g_{2m+1}\gamma), \cdots$$

Since the successive first members make up a decreasing sequence of positive integers so long as the corresponding second members are positive, we must reach an m such that

$$g_{2m+1}\gamma > g_{2m}\alpha \quad \text{and} \quad g_{2m+3}\gamma < g_{2m+2}\alpha .$$

Then

$$(\alpha_0, \gamma_0) = (g_{2m-1}\alpha - g_{2m}\gamma, -g_{2m}\alpha + g_{2m+1}\gamma)$$

satisfies

$$\alpha_0 > (5/2)\gamma_0 ,$$

and exactly one of (α_0, γ_0) and

$$(\alpha_1, \gamma_1) = (5\alpha_0 - 12\gamma_0, 2\alpha_0 - 5\gamma_0)$$

satisfies (2) with $p = 2$.

The transformation which takes (α_0, γ_0) to (α_1, γ_1) has determinant -1 and (α_0, γ_0), (α_1, γ_1) are associated with different minimum roots of (11). Thus the last statement of the lemma is easily verified.

5. PROOF OF THEOREM 2

Lemma 8. Let $(c, m) = 1$. Then

$$x^2 \equiv c \mod m$$

has 2^{r+w} roots if it has any roots, where r is the number of distinct odd primes dividing m and w is given by

$$w = \begin{cases} 0 & \text{if 4 does not divide } m \\ 1 & \text{if 4 but not 8 divides } m \\ 2 & \text{if 8 divides } m . \end{cases}$$

Proof. This is a well-known result ([2, p. 75, Th. 60]).

For $p = 1, 2$, let r be the number of distinct odd primes dividing $4D/(d, 4D)$. It is easy to verify using Lemma 8 that the congruences (8) and (11) have 2^{r+1} roots. Then Theorem 2 follows from Lemmas 3, 5, and 7.

6. CONCLUDING REMARKS

We comment briefly on the reasons for confining detailed discussion above to the cases $p = 1, 2$.

Let $h(d)$ be the number of distinct non-equivalent reduced forms of discriminant d. We can make little progress if $h(d) > 1$, because for such d the problem of determining all positive integers properly represented by $(1, -p, -1)$ even without the restriction (2) is unsolved. We remark that $h(d) = 1$ for $p = 1, 2, 3, 5, 7$, but $h(d) = 2$ for $p = 4, 6$.

However, it is not enough simply to confine ourselves to the study of those p for which $h(d) = 1$. We have seen that for $p = 1, 2$ the converse of Lemma 1 Corollary is true and for any properly representable D a proper representable D a proper representation satisfying (2) can be found. However, for $p \geq 3$ there exist integers D which are properly represented by $(1, -p, -1)$ but which have no proper representation satisfying (2), and it is not simple to describe the subset of S_p composed of such integers.

REFERENCES

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