ELEMENTARY PROBLEMS AND SOLUTIONS

Edited By
A. P. HILLMAN
University of New Mexico, Albuquerque, New Mexico

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico, 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets in the format used below. Solutions should be received within three months of the publication date.

B-136 Proposed by Phil Mana, University of New Mexico, Albuquerque, N. Mex.

Let P_n be the n^{th} Pell number defined by $P_1 = 1$, $P_2 = 2$, and $P_{n+2} = 2P_{n+1} + P_n$. Show that $P^2_{n+1} + P^2_n = P_{2n+1}$.

B-137 Proposed by Phil Mana, University of New Mexico, Albuquerque, N. Mex.

Let P_n be the n^{th} Pell number. Show that $P_{2n+1} + P_{2n} = 2P^2_{n+1} - 2P^2_n - (-1)^n$.

B-138 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Show that for any non-negative integer k and any integer $n > 1$ there is an n by n matrix with integral entries whose top row is $F_{k+1}, F_{k+2}, \cdots, F_{k+n}$ and whose determinant is 1.

B-139 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Show that the sequence 1, 1, 1, 1, 4, 4, 9, 9, 25, 25, $\cdots$ defined by $a_{2n-1} = a_{2n} = F^2_n$ is complete even if an a_j with $j \le 6$ is omitted but that the sequence is not complete if an a_j with $j \ge 7$ is omitted.

B-140 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Show that $F_{ab} > F_aF_b$ if a and b are integers greater than 1.

B-141 Proposed by Charles R. Wall, University of Tennessee, Knoxville, Tenn.

Show that no Fibonacci number F_n nor Lucas number L_n is an even perfect number.

SOLUTIONS

ERRATUM

A line was omitted in the printing of the solution of B-95 in Vol. 5, No. 2, p. 204. The submitted solution follows:

Solution by Charles W. Trigg, San Diego, California.

For $n \geq 3$, F_k is divisible by 2^n if k is of the form $2^{n-2} \cdot 3(1+2m)$, $m = 0, 1, 2, \cdots$. If k is of the form $3(1+2m)$, F_k is divisible by 2 but by no higher power of 2. Hence, the highest power of 2 that exactly divides $F_1F_2F_3 \cdots F_{100}$ is

$$[(100-3)/6+1] + 3[(100+6)/12] + 4[112/24] + 5[124/48] + 6[148/96] + 7[196/192]$$

or 80. As usual, $[x]$ indicates the largest integer in x.

A PARTIAL SUM INEQUALITY

B-118 Proposed by J. L. Brown, Jr., Pennsylvania State University, State College, Pennsylvania.

Let $F_1 = 1 = F_2$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$. Show that for all $n \geq 1$ that

$$\sum_{k=1}^{n} (F_k/2^k) < 2 .$$

Solution by Douglas Lind, University of Virginia, Charlottesville, Virginia.

From the standard generating function

$$\sum_{k=1}^{\infty} F_k x^k = \frac{x}{1 - x - x^2}$$

which converges for $|x| < 2/(1+\sqrt{5})$, we find for $x = \frac{1}{2}$ that

$$\sum_{k=1}^{\infty} (F_k/2^k) = 2 ,$$

from which the result follows.

Also solved by E. M. Clark, Lawrence D. Gould, John Ivie, Clifford Juhlke, Bruce W. King, Geoffrey Lee, Robert L. Mercer, F. D. Parker, C. B. A. Peck, J. Ramanna, A. C. Shannon, John Wessner, David Zeitlin, and the proposer.

A FIBONACCI TRAPEZOID

B-119 Proposed by Jim Woolum, Clayton Valley High School, Concord, Calif.

What is the area of an equilateral trapezoid whose bases are F_{n-1} and F_{n+1} and whose lateral side is F_n ?

Solution by F. D. Parker, St. Lawrence University, Canton, New York.

The difference between the two bases is F_n, so that the base angles of the trapezoid are $\pi/3$, and the altitude is $F_n\sqrt{3}/2$. Thus the area is given by $\sqrt{3}F_n(F_{n+1} + F_{n-1})/4$. This result can be simplified to $\sqrt{3}F_{2n}/4$.

Also solved by Herta T. Freitag, Lawrence D. Gould, J. A. H. Hunter, John Ivie, Bruce W. King, Geoffrey Lee, Douglas Lind, John W. Milsom, C. B. A. Peck, A. C. Shannon, John Wessner, and the proposer.

A TRIANGULAR NUMBERS RELATION

B-120 Proposed by Phil Mana, University of New Mexico, Albuquerque, N. Mex.

Find a simple function g such that g(n) is an integer when n is an integer and $g(m + n) - g(m) - g(n) = mn$.

Solution by J. A. H. Hunter, Toronto, Canada.

Taking suitable integral values for m and n we find that:

$$g(2) = 2g(1) + 1 \qquad g(3) = 3g(1) + 3$$
$$g(4) = 4g(1) + 6 \qquad g(5) = 5g(1) + 10$$

The sequence 1, 3, 6, 10 suggests triangular numbers. Hence, taking $g(1) = 1$, we have: $g(n) = n(n + 1)/2$. But g(1) may be any positive integer, so we take the generak function:

$$g(m) = m(m + 2k + 1)/2, \qquad k \text{ an integer.}$$

Also solved by L. Carlitz, E. M. Clarke, Douglas Lind, C. B. A. Peck, J. Ramanna, David Zeitlin, and the proposer.

A CONGRUENCE MODULO F_d

B-121 Proposed by Phil Mana, University of New Mexico, Albuquerque, N. Mex.

Let n, q, d and r be integers with $n \geq 0$, $d > 0$, $n = qd + r$, and $0 \leq r < d$. Prove that

$$F_n \equiv F_{d+1})^q F_r \pmod{F_d}.$$

Solution by Douglas Lind, University of Virginia, Charlottesville, Virginia.

Vinson ["The Relation of the Period Modulo m to the Rank of Apparition of m in the Fibonacci Sequence," Fibonacci Quarterly, 1(1963), No. 2, 37-45] has shown that

$$F_n = F_{qd+r} = \sum_{j=0}^{q} \binom{q}{j} F_d^j F_{d-1}^{q-j} F_{r+j} \quad (q \geq 0) .$$

Hence

$$F_n \equiv F_{d-1}^q F_r \equiv F_{d+1}^q F_r \quad (\text{mod } F_d)$$

since

$$F_{d+1} \equiv F_{d-1} \quad (\text{mod } F_d) .$$

Also solved by David L. Estrin and the proposer.

ANALOG OF A MULTIPLE ANGLE FORMULA

B-122 Proposed by A. J. Montleaf, University of New Mexico, Albuquerque, N.M.

Show that

$$\sin[(2k+1)\theta]/\sin\theta = 2\cos[2k\theta] + 2\cos[2(k-1)\theta] + 2\cos[2(k-2)\theta] + \cdots + \\ + 2\cos[2\theta] + 1$$

and obtain the analogous formula for $F_{(2k+1)m}/F_m$ in terms of Lucas numbers.

Solution by Paul A. Anderson, University of Minnesota, Minneapolis, Minn.

The stated formula is well known (see, for example, Taylor, <u>Advanced Calculus</u>, p. 729). The analogous formula for $F_{(2k+1)m}/F_m$ is

$$\frac{F_{(2k+1)m}}{F_m} = (-1)^{km} + (-1)^{(k+1)m} L_{2m} + (-1)^{(k+2)m} L_{4m} + \cdots + (-1)^{2km} L_{2km} .$$

The proof is by induction on k. For k = 1,

$$\frac{F_{3m}}{F_m} = \frac{a^{3m} - b^{3m}}{a^m - b^m} = a^{2m} + (ab)^m + b^{2m} = L_{2m} + (-1)^m,$$

where

$$a = \frac{1 + \sqrt{5}}{2}, \quad b = \frac{1 - \sqrt{5}}{2} .$$

If

$$\frac{F_{(2k+1)m}}{F_m} = (-1)^{km} + \sum_{i=1}^{k} (-1)^{(k+i)m} L_{2mi} ,$$

then

$$L_{(2k+2)m} + (-1)^m \frac{F_{(2k+1)m}}{F_m} = a^{(2k+2)m} + b^{(2k+2)m} + (-1)^m \frac{a^{(2k+1)m} - b^{(2k+1)m}}{a^m - b^m} =$$

$$\frac{a^{(2k+3)m} - b^{(2k+3)m} + a^m b^{(2k+2)m} - a^{(2k+2)m} b^m + (-1)^m (a^{(2k+1)m} - b^{(2k+1)m})}{a^m - b^m} =$$

$$\frac{a^{(2k+3)m} - b^{(2k+3)m}}{a^m - b^m} = \frac{F_{(2k+3)m}}{F_m} .$$

Also solved by A. C. Shannon and the proposer.

SQUARE SUM OF SUCCESSIVE SQUARES

B-123 (From B-102, Proposed by G. L. Alexanderson, University of Santa Clara, Santa Clara, California)

Show that all the positive integral solutions of $x^2 + (x \pm 1)^2 = z^2$ are given by

$$x_n = (P_{n+1})^2 - (P_n)^2 ; \quad z_n = (P_{n+1})^2 + (P_n)^2; \quad n = 1, 2, \cdots ;$$

where P_n is the Pell number defined by $P_1 = 1$, $P_2 = 2$, and $P_{n+2} = 2P_{n+1} + P_n$.

Solution by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Letting $w = 2x \pm 1$ changes $x^2 + (x \pm 1)^2 = z^2$ into $w^2 - 2z^2 = -1$. Let Z be the ring of the integers and let $Z\sqrt{2}$ be the ring consisting of the real numbers $\alpha = z + b\sqrt{2}$ with a and b in Z. Let V consist of the positive real numbers $\alpha = a + b\sqrt{2}$ of $Z[\sqrt{2}]$ such that $a^2 - 2b^2 = -1$. Then V can be shown to be a group under multiplication. Since V has no number between 1 and $1 + \sqrt{2}$, it follows that V is the cyclic group generated by $1 + \sqrt{2}$. The odd powers $(1 + \sqrt{2})^{2n-1}$ lead to $a^2 - 2b^2 = -1$. Therefore the positive integral solutions of $w^2 - 2z^2 = -1$ are obtained by equating "rational" and "irrational" parts of $w_n + z_n\sqrt{2} = (1 + \sqrt{2})^{2n-1}$, i.e.,

$$w_n = [(1 + \sqrt{2})^{2n-1} + (1 - \sqrt{2})^{2n-1}]/2, \quad z_n = [(1 + \sqrt{2})^{2n-1} - (1 - \sqrt{2})^{2n-1}]/2\sqrt{2}.$$

The desired formulas then may be found using the analogue $P_n = [(1 + \sqrt{2})^n - (1 - \sqrt{2})^n]/2\sqrt{2}$ of one of the Binet formulas.

Also solved by A. C. Shannon and the proposer.

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(Continued from p. 176)

$$P_3(x) = 32 - 13x - 99x^2 - 32x^3 + 9x^4 + x^5$$
$$P_4(x) = 243 + 1181x - 1952x^2 - 1271x^3 + 257x^4 + 32x^5$$
$$P_5(x) = 3125 + 7768x - 15851x^2 - 9752x^3 + 1944x^4 + 243x^5$$

$$\sum_{n=0}^{\infty} F_{n+k}^6 x^n = \frac{P_k(x)}{1 - 13x - 104x^2 + 260x^3 + 260x^4 - 104x^5 - 13x^6 + x^7}$$

$$k = 0, 1, 2, 3, 4, 5, 6$$

$$P_0(x) = x(1 - 12x - 53x^2 + 53x^3 + 12x^4 - x^5)$$
$$P_1(x) = 1 - 12x - 53x^2 + 53x^3 + 12x^4 - x^5$$
$$P_2(x) = 1 + 51x - 207x^2 - 248x^3 + 103x^4 + 13x^5 - x^6$$
$$P_3(x) = 64 - 103x - 508x^2 - 157x^3 + 117x^4 + 12x^5 - x^6$$
$$P_4(x) = 729 + 6148x - 16{,}797x^2 - 16{,}523x^3 + 6{,}668x^4 + 831x^5 - 64x^6$$
$$P_5(x) = 15{,}625 + 59{,}019x - 206{,}063x^2 - 182{,}872x^3 + 76{,}644x^4 + 9413x^5 - 729x^6$$
$$P_6(x) = 262{,}144 + 1{,}418{,}937x - 4{,}245{,}372x^2 - 3{,}985{,}856x^3 + 1{,}634{,}413x^4 + 202{,}396x^5 - 15{,}625x^6$$

(Continued on p. 166.)