ON THE GENERALIZED LANGFORD PROBLEM

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For n a positive integer, the sequence $a_1, \cdots, a_{2n}$ is said to be a perfect sequence for n if (a) each integer i in the range $1 \le i \le n$ appears exactly twice in the sequence, and (b) the double occurrence of i in the sequence is separated by exactly i entries. Thus 4 1 3 1 2 4 3 2 is a perfect sequence for $n = 4$. The problem of determining all integers n having a perfect sequence is posed in [1] and resolved in [2] and [3]. In particular, n has an associated perfect sequence if and only if $n \equiv 3$ or $4 \pmod 4$.

In [4], the problem is generalized by introducing the notion of a perfect s-sequence for an integer n. Namely, a perfect s-sequence for n (with s, $n > 0$) is a sequence of length sn such that (a) each of the integers $1, 2, \cdots, n$ occurs exactly s times in the sequence and (b) between any two consecutive occurrences of the integer i there are exactly i entries. The problem of determining all s and n for which there are perfect s-sequences is then posed in [4]. (The existence of a perfect s-sequence for any n with $s > 2$ is yet in doubt.) It is shown in [4] that no perfect 3-sequences exist for $n = 2$, 3, 4, 5, and 6.

The following theorems expand upon the above results pertaining to the non-existence of perfect s-sequences for various classes of n and s.

Theorem 1. Let $s = 2t$. Then there is no generalized s-sequence for $n \equiv 1$ or $2 \pmod 4$.

Proof. Let p_i denote the position of the first occurrence of the integer i $(1 \le i \le n)$ in the sequence. The integer i then occurs in positions p_i, $p_i + (i + 1), \cdots, p_i + (s - 1)(i + 1)$. The sn integers $p_i + j(i + 1)$ (with $i = 1, \cdots, n$; $j = 0, 1, \cdots, s - 1$) are however the integers $1, \cdots, sn$ in some order. Thus

$$\sum_{i=1}^{n} \sum_{j=0}^{s-1} \{p_i + j(i + 1)\} = \sum_{k=1}^{sn} k \quad .$$

Letting

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$$P = \sum_{i=1}^{n} p_i ,$$

the latter equality yields

$$sP + \frac{(s-1)s}{2}\left\{\frac{(n+1)(n+2)}{2} - 1\right\} = \frac{sn(sn+1)}{2}$$

or

$$P = \frac{n\{(s+1)n - (3s-5)\}}{4}$$

Inasmuch as P is an integer, the numerator $N = n\{(s+1)n - (3s-5)\}$ must be divisible by 4. But for $n \equiv 1 \pmod 4$,

$$N \equiv (s+1) - (3s-5) \equiv -4t + 6 \equiv 2 \pmod 4 ,$$

where $s = 2t$, which is impossible. Similarly, for $n \equiv 2 \pmod 4$,

$$N \equiv 2\{2(s+1) - (3s-5)\} \equiv -4t + 14 \equiv 2 \pmod 4$$

which is also impossible.

We now extend the results in [4] by proving there is no 3-sequence for $n \equiv 2, 3, 4, 5, 6,$ or $7 \pmod 9$. Actually we show somewhat more in the next theorem.

<u>Theorem 2.</u> Let $s = 6r + 3$ (with $r \geq 0$). Then there is no perfect s-sequence for any $n \equiv 2, 3, 4, 5, 6,$ or $7 \pmod 9$.

<u>Proof.</u> Let q_i denote the position that integer i occurs for the $(3r + 2)^{th}$ time (i. e., q_i is the position of the "middle" occurrence of i). Then i occurs in positions $q_i + j(i+1)$ for $j = -2(2r+1), -3r, \cdots, 3r, (3r+1)$. The sn integers $q_i + j(i+1)$ (with $i = 1, \cdots, n;\ j = -(3r+1), \cdots, 3r+1$) are then the integers $1, 2, 3, \cdots, sn$ in some order. Thus

$$\sum_{i=1}^{n} \sum_{j=-(3r+1)}^{3r+1} \{q_i + j(i+1)\}^2 = \sum_{k=1}^{sn} k^2 .$$

Letting

$$Q = \sum_{i=1}^{n} q_i^2 ,$$

and noting that the linear terms on the left-hand side of the last equation cancel, we have

$$sQ + 2\left\{\frac{(3r+1)(3r+2)s}{6}\right\}\left\{\frac{(n+1)(n+2)(2n+3)}{6} - 1\right\}$$
$$= \frac{sn(sn+1)(2sn+1)}{6}$$

Cancelling out s and collecting terms yields $Q = M/18$, where the numerator M is given by

$$M = (198r^2 + 198r + 50)n^3 - (81r^2 + 27r - 9)n^2 - (117r^2 + 117r + 23)n .$$

Inasmuch as Q is an integer, the numerator M must be divisible by 9. But

$$M \equiv 50n^3 - 23n \equiv 5(n^3 - n) \pmod 9 .$$

It is easily verified from the latter that for the values of n under consideration, namely, $n \equiv 2, 3, 4, 5, 6,$ or $7 \pmod 9$ we have $M \equiv 3$ or $6 \pmod 9$. Thus M is not divisible by 9 which provides a contradiction.

REFERENCES

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FIBONACCIAN ILLUSTRATION OF L'HOSPITAL'S RULE

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In [1] there is the statement: using the convention $F_0/F_0 = 1.$" $[F_n = F_{n+1} + F_{n-2},\ F_0 = 0, F_1 = 1]$.

In this note it will be shown how the equation $F_0/F_0 = 1$ follows naturally from L'Hospital's Rule applied to the continuous function

$$F_x \equiv \frac{1}{\sqrt{5}}\left(\phi^x - \phi^{-x}\cos \pi x\right) \qquad [\phi = 2^{-1}(1+\sqrt{5})] .$$

F_x obviously reduces to the Fibonacci numbers F_n when $n = 0, \pm 1, \pm 2, \pm 3, \cdots$. Then

$$\frac{F_0}{F_0} = \left.\frac{\frac{1}{\sqrt{5}}(\phi^x - \phi^{-x}\cos \pi x)}{\frac{1}{\sqrt{5}}(\phi^x - \phi^{-x}\cos \pi x)}\right]_{x=0} = \left.\frac{\frac{d}{dx}(\phi^x - \phi^{-x}\cos \pi x)}{\frac{d}{dx}(\phi^x - \phi^{-x}\cos \pi x)}\right]_{x=0}$$

$$= \left.\frac{(\log \phi)\phi^x - (\log \phi^{-1})\phi^{-x}\cos \pi x + \phi^{-x}\pi \sin \pi x}{(\log \phi)\phi^x - (\log \phi^{-1})\phi^{-x}\cos \pi x + \phi^{-x}\pi \sin \pi x}\right]_{x=0}$$

$$= \frac{\log \phi - \log \phi^{-1}}{\log \phi - \log \phi^{-1}} = 1 .$$

(Continued on p. 150.)