FURTHER PROPERTIES OF MORGAN-VOYCE POLYNOMIALS

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1. INTRODUCTION

A set of polynomials $B_n(x)$ and $b_n(x)$ were first defined by Morgan-Voyce [1] as,

(1) $$b_n(x) = x B_{n-1}(x) + b_{n-1}(x) \qquad (n \geq 1)$$

(2) $$B_n(x) = (x+1)B_{n-1}(x) + b_{n-1}(x) \qquad (n \geq 1)$$

with

(3) $$b_0(x) = B_0(x) = 1.$$

In an earlier article [2], a number of properties of these polynomials $B_n(x)$ and $b_n(x)$ were derived and these were used in a later article to establish some interesting Fibonacci identities [3]. We shall now consider some further properties of these polynomials and establish their relations with the Fibonacci polynomials $f_n(x)$.

2. GENERATING MATRIX

The matrix Q defined by,

(4) $$Q = \begin{bmatrix} (x+2) & -1 \\ 1 & 0 \end{bmatrix}$$

may be called as the generating matrix, since we may establish by induction that,

(5) $$Q^n = \begin{bmatrix} B_n & -B_{n-1} \\ B_{n-1} & -B_{n-2} \end{bmatrix}$$

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Hence,

$$\begin{bmatrix} b_n & -b_{n-1} \\ b_{n-1} & -b_{n-2} \end{bmatrix} = \begin{bmatrix} (B_n - B_{n-1}) & -(B_{n-1} - B_{n-2}) \\ (B_{n-1} - B_{n-2}) & -(B_{n-2} - B_{n-3}) \end{bmatrix} = Q^n - Q^{n-1}$$

$$= Q^{n-1}(Q - I) \tag{6}$$

Since the determinant of $Q = 1$, we have

$$B_{n+1}B_{n-1} - B_n^2 = -1 \tag{7}$$

and

$$\begin{vmatrix} b_n & -b_{n-1} \\ b_{n-1} & -b_{n-2} \end{vmatrix} = |Q - I| = \begin{vmatrix} x+1 & -1 \\ 1 & -1 \end{vmatrix} = x$$

or

$$b_{n+1}b_{n-1} - b_n^2 = x \,. \tag{8}$$

3. B_n AND b_n AS TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

Letting $\cos\theta = (x + 2)/2$ in the identity

$$\sin(n + 1)\theta + \sin(n - 1)\theta = 2\sin(n\theta)\cos\theta$$

we have

$$\frac{\sin(n+1)\theta}{\sin\theta} + \frac{\sin(n-1)\theta}{\sin\theta} = (x+2)\frac{\sin n\theta}{\sin\theta} \qquad (-4 \le x \le 0)\,,$$

with

$$\frac{\sin(n+1)\theta}{\sin\theta} = 1 \qquad \text{for } n = 0$$

$$= (x + 2) \qquad \text{for } n = 1\,.$$

Thus,

$$\frac{\sin (n+1)\theta}{\sin \theta}$$

satisfies the difference equation for B_n. Hence,

(9) $$B_n(x) = \frac{\sin (n+1)\theta}{\sin \theta} \qquad (-4 \le x \le 0)$$

Similarly, if $\cosh \phi = (x+2)/2$, then

(10) $$B_n(x) = \frac{\sinh (n+1)\phi}{\sinh \phi} \qquad (x \ge 0)$$

Since $b_n = B_n - B_{n-1}$, we have

(11a) $$b_n(x) = \frac{\cos (2n+1)\theta/2}{\cos \theta/2} \qquad (-4 \le x \le 0)$$

and

(11b) $$b_n(x) = \frac{\cosh (2n+1)\phi/2}{\cosh \phi/2} \qquad (x \ge 0)$$

4. DIFFERENTIAL EQUATIONS FOR $B_n(x)$ AND $b_n(x)$

It has been shown earlier [2] that

(12) $$B_n(x) = \sum_{k=0}^{n} \binom{n+k-1}{n-k} x^k = \sum_{k=0}^{n} c_n^k x^k$$

and

(13) $$b_n(x) = \sum_{k=0}^{n} \binom{n+k}{n-k} x^k = \sum_{k=0}^{n} d_n^k x^k .$$

Hence

$$\frac{c_n^{k+1}}{c_n^k} = \frac{\binom{n+k+2}{n-k-1}}{\binom{n+k+1}{n-k}} = \frac{(n-k)(n+k+2)}{(2k+3)(2k+2)}$$

Thus, the coefficients of x^k and x^{k+1} of $B_n(x)$ are related by

$$\text{(14)} \qquad k(k-1)c_n^k + 4(k+1)k\, c_n^{k+1} + 3k\, c_n^k + 6(k+1)c_n^{k+1} - n(n+2)c_n^k = 0 \cdots$$

But the coefficient of x^k in the expansion of

$$x^2B_n'' + 4\,x\,B_n' + 3x\,B_n' + 6\,B_n' - n(n+2)B_n$$

is the same as the left-hand side expression of (14). Hence, $B_n(x)$ satisfies the differential equation

$$\text{(15)} \qquad x(x+4)y'' + 3(x+2)y' - n(n+2)y = 0$$

Similarly, starting with (13) we can show that $b_n(x)$ satisfies the differential equation

$$\text{(16)} \qquad x(x+4)y'' + 2(x+1)y' - n(n+1)y = 0$$

Using (15) and (16) we shall now derive some identities for $B_n(x)$ and $b_n(x)$. We have from (15)

$$x(x+4)(B_n'' - B_{n-1}'') + 3(x+2)(B_n' - B_{n-1}') - n(n+2)B_n + (n+1)(n-1)B_{n-1} =$$

or,

$$x(x+4)b_n'' + 3(x+2)b_n' - n(n+1)b_n - nB_n - (n+1)B_{n-1} = 0 .$$

Using (16) this may be reduced to

$$(17) \qquad (x + 4)b_n'(x) = nB_n(x) + (n + 1)B_{n-1}(x) .$$

Hence,

$$(18) \qquad (x + 4)(b_{n+1}' - b_n') = (n + 1)B_{n+1} + (n + 2)B_n - nB_n - (n + 1)B_{n-1}$$

Differentiating (1) we get,

$$(19) \qquad b_{n+1}' - b_n' = x B_n' + B_n$$

Substituting (19) in (18) and simplifying we have

$$(20) \qquad x(x + 4)B_n'(x) = nB_{n+1}(x) - (n + 2)B_{n-1}(x)$$

From (20) we may derive that

$$(21) \qquad x(x + 4)b_n'(x) = nb_{n+1}(x) + b_n(x) - (n + 1)b_{n-1}(x) .$$

5. INTEGRAL PROPERTIES

It has been shown earlier [2] that,

$$(22) \qquad \int b_n(x)dx = \frac{B_{n+1}(x) - B_{n-1}(x)}{(n + 1)} + c$$

c being an arbitrary constant. We also know that,

$$(23) \qquad \begin{aligned} B_n(0) &= (n + 1); \quad B_n(-4) = (-1)^n(n + 1) \\ b_n(0) &= 1 \quad ; \quad b_n(-4) = (-1)^n(2n + 1) \end{aligned}$$

Hence, from (22) and (23) we have the two special integrals,

$$(24a) \qquad \int_{-4}^{0} B_{2n}(x)\, dx = 4/(2n + 1)$$

and

$$\int_{-4}^{0} B_{2n+1}(x)\, dx = 0 \tag{24b}$$

Since

$$B_n^2(x) = \sum_0^n B_{2m}$$

we have

$$\int_{-4}^{0} B_n^2(x)\, dx = \sum_0^n 4/(2m+1) \tag{25}$$

Similarly, the following integrals may be established:

$$\int_{-4}^{0} b_n^2(x)\, dx = -\int_{-4}^{0} b_{2n+1}(x)\, dx = 4/(2n+1)$$

$$\int_{-4}^{0} B_n(x)B_{n+1}(x)\, dx = 0$$

$$\int_{-4}^{0} b_n(x)\, B_n(x)\, dx = -\int_{-4}^{0} b_{n+1}(x)B_n(x)dx = -4\sum_0^n 1/(2m+1)$$

$$\int_{-4}^{0} b_n(x)b_{n+1}(x)\, dx = -4 - 8\sum_1^n 1(2m+1)$$

$$\int_{-4}^{0} B_{n+1}(x)\, B_{n-1}(x)\, dx = 4 \sum_{1}^{n} 1/(2m+1)$$

$$\int_{-4}^{0} b_{n+1}(x)\, b_{n-1}(x)\, dx = 8 \sum_{1}^{n-1} 1/(2m+1) + 4/(2n+1) - 8$$

$$\int_{-4}^{0} b_n^2(x)\, dx = 8 \sum_{1}^{n-1} 1/(2m+1) + 4/(2n+1) .$$

6. ZEROS OF $B_n(x)$ AND $b_n(x)$

From (9) we see that the zeros of $B_n(x)$ are given by $\sin (n + 1)\theta = 0$. Hence,

$$\theta = (r\pi)/(n+1), \quad r = 1, 2, \cdots, n .$$

Therefore,

$$(x + 2) = 2 \cos \frac{r}{n+1} \pi$$

or,

$$x = -4 \sin^2 \left\{ \frac{r}{n+1} \cdot \frac{\pi}{2} \right\}, \quad r = 1, 2, \cdots, n .$$

Similarly, the zeros of $b_n(x)$ are given by

$$-4 \sin^2 \left\{ \frac{2r-1}{2r+1} \cdot \frac{\pi}{2} \right\}, \quad r = 1, 2, \cdots, n.$$

Thus the zeros of $B_n(x)$ and $b_n(x)$ are real, negative and distinct.

7. $B_n(x)$, $b_n(x)$ AND $f_n(x)$

The Fibonacci polynomials $f_n(x)$ are defined by

(26) $$f_{n+1}(x) = x f_n(x) + f_{n-1}(x) \qquad (n \geq 2)$$

with

$$f_1(x) = 1 \quad \text{and} \quad f_2(x) = x .$$

It is also known [4] that

(27) $$f_n(x) = \sum_{j=0}^{[(n-1)/2]} \binom{n-j-1}{j} x^{n-2j-1} ,$$

where $[n/2]$ is the greatest integer in $(n/2)$. Hence

$$f_{2n+1}(x) = \sum_{j=0}^{n} \binom{2n-j}{j} x^{2n-2j} = \sum_{r=0}^{n} \binom{n+r}{n-r} (x^2)^r$$

$$= b_n(x^2) ,$$

from (13). Hence,

(28) $$b_n(x^2) = f_{2n+1}(x) .$$

Now

$$f_{2n+3}(x) - f_{2n+1}(x) = xf_{2n+2}(x)$$

or

$$b_{n+1}(x^2) - b_n(x^2) = xf_{2n+2}(x)$$

Hence from (1) we have

$$x^2 B_n(x^2) = x f_{2n+2}(x)$$

or

$$B_n(x^2) = \frac{1}{x} f_{2n+2}(x) \tag{29}$$

Thus, $B_n(x)$, $b_n(x)$ and $f_n(x)$ are interrelated.

(See also H-73 Oct. 1967 pp 255-56)

REFERENCES

1. A. M. Morgan-Voyce, "Ladder Network Analysis Using Fibonacci Numbers," IRE. Transactions on Circuit Theory, Vol. CT-6, Sept. 1959, pp. 321-322.

2. M. N. S. Swamy, "Properties of the Polynomials Defined by Morgan-Voyce," Fibonacci Quarterly, Vol. 4, Feb. 1966, pp. 73-81.

3. M. N. S. Swamy, "More Fibonacci Identities," Fibonacci Quarterly, Vol. 4, Dec. 1966, pp. 369-372.

4. M. N. S. Swamy, Problem B-74, Fibonacci Quarterly, Vol. 3, Oct. 1965, p. 236.

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(Continued from p. 161.)

(Compare this problem with H-65 and above solution formula with the formula

$$\frac{2x}{1 - 4x - x^2} = \sum_{n=0}^{\infty} F_{3n} x^n$$

in the Fibonacci Quarterly, Vol. 2, No. 3, p. 208.)

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