ON Q-FIBONACCI POLYNOMIALS

SELMO TAUBER
Portland State College, Portland, Oregon

INTRODUCTION

Throughout this paper we shall use the following notation:

$$\sum_{s_1=a_1}^{b_1} \sum_{s_2=a_2}^{b_2} \cdots \sum_{s_n=a_n}^{b_n} = \left(\Sigma^{(n)}, \; s_1 \Big|_{a_1}^{b_1}, \quad s_2 \Big|_{a_2}^{b_2}, \cdots s_n \Big|_{a_n}^{b_n} \right) .$$

Let $F_0, F_1, F_2, \cdots, F_n, \cdots$ be the sequence of Fibonacci numbers, i.e., 0, 1, 1, 2, 3, 5, 8, $\cdots$. According to [1] we define $n, m, k \geq 0$.

(1) $$Q(x;1,-F,n) = \eta(x,k,n) = \prod_{m=1}^{n} (1 - xF_{k+m}) = \sum_{s=0}^{n} A(k,n,s)x^s ,$$

(2) $$\eta(x,k,0) = 1 ,$$

(3) $$x^n = \sum_{m=0}^{n} B(k,n,m)\eta(x,k,m)$$

(4) $$1 = B(k,0,0)\eta(x,k,0) ,$$

(5) $$A(k,n,s), \quad B(k,n,m) = 0 \quad \text{for} \quad n < m, \; n < 0, \; m < 0 .$$

The A and B numbers are quasi-orthogonal. (For a set of comprehensive definitions of orthogonality and quasi-orthogonality cf. [3].) Thus

(Received February 1967)

(6) $$\sum_{s=m}^{n} B(k,n,s)A(k,s,m) = \delta_n^m ,$$

where δ_n^m is the Kronecker Delta.

Still according to [1] the A and B numbers satisfy the difference equations

(7) $$A(k,n,m) = A(k,n-1,m) - F_{n+k}A(k,n-1,m-1)$$

(8) $$B(k,n,m) = (F_{m+1+k})^{-1}B(k,n-1,m) - (F_{m+k})^{-1}B(k,n-1,m-1) ,$$

where the error in Eqs. (10) and (12) of [1] has been corrected.

2. BASIC RELATIONS

According to the preceding definitions we can write

$$\eta(x,k,n) = \prod_{m=1}^{n}(1 - xF_{k+m}) = \prod_{m=1}^{p}(1 - xF_{k+m})\prod_{m=p+1}^{n}(1 - xF_{k+m})$$

$$= \eta(x,k,p)\prod_{m=p+1}^{n}(1 - xF_{k+m}) .$$

In the last product we take $m - p = s$, $m = s + p$, so that for $m = p + 1$, $s = 1$, and for $m = n$, $s = n - p$, thus

$$\prod_{m=p+1}^{n}(1 - xF_{k+m}) = \prod_{s=1}^{n-p}(1 - xF_{k+p+s}) = \eta(x,k+p,n-p),$$

i.e.,

(9) $$\eta(x,k,n) = \eta(x,k,p)\eta(x,k+p,n-p) ,$$

or,

$$(10) \qquad \eta(x, k, n+p) = \eta(x, k, p)\eta(x, k+p, n) .$$

By substitution into (10) of the polynomial form for the η's we obtain

$$(11) \qquad \sum_{m=0}^{n} A(k, n+p, m)x^m = \left[\sum_{s=0}^{p} A(k, p, s)x^s\right]\left[\sum_{t=0}^{n} A(k+p, n, t)x^t\right] ,$$

so that equating the coefficients of same powers of x we have with $s + t = m$,

$$(12) \qquad A(k, n+p, m) = \sum_{s=0}^{m} A(k, p, s)A(k+p, n, m-s)$$

which is a convolution formula for the A numbers. Also

$$x^n = \sum_{m=0}^{n} B(k, n, m)\eta(x, k, m) , \quad x^p = \sum_{s=0}^{p} B(k+p, p, s)\eta(x, k+p, s) ,$$

hence,

$$\begin{aligned} x^{n+p} &= \sum_{t=0}^{n+p} B(k, n+p, t)\eta(x, k, t) \\ &= \left[\sum_{m=0}^{n} B(k, n, m)\eta(x, k, m)\right]\left[\sum_{s=0}^{p} B(k+p, p, s)\eta(x, k+p, s)\right] \\ &= \left(\sum^{(2)}, m\Big|_0^n, s\Big|_0^p\right) B(k, n, m)B(k+p, p, s)\eta(x, k, m)\eta(x, k+p, s) . \end{aligned}$$

By comparing the coefficients of $\eta(x, k, t)$ and using (10) with $m + s = t$ we obtain

$$B(k, n + p, t) = \sum_{m=0}^{t} B(k, n, m) B(k + p, p, t - m) \quad , \tag{13}$$

which is a convolution formula for the B numbers.

3. LAH TYPE RELATIONS

According to [2] we have for $k \neq h$

$$\sum_{s=m}^{n} A(k, n, s) B(h, s, m) = L(k, h, n, m) \tag{14}$$

$$\sum_{s=m}^{n} A(h, n, s) B(k, s, m) = L(h, k, n, m) \tag{15}$$

$$\eta(x, j, n) = \sum_{m=0}^{n} \eta(x, i, m) L(j, i, n, m) \quad , \tag{16}$$

where $k, h = i, j$, with $i \neq j$. Again according to [2] there is a quasi-orthogonality relation between the Lah numbers:

$$\sum_{s=m}^{n} L(i, j, n, s) L(j, i, s, m) = \delta_n^m \quad . \tag{17}$$

Still according to [2] the recurrence relation for Lah numbers is

(18)
$$L(i,j,n,m) = \left[1 - (F_{j+n}/F_{i+m+1})\right]L(i,j,n-1,m) + (F_{j+n}/F_{i+m})L(i,j,n-1,m-1) .$$

4. GENERALIZATION TO THREE VARIABLES

Although we could generalize to p variables we prefer to limit ourselves to $p = 3$ for the sake of simplicity. Let

(19)
$$\eta(x,y,z;\,k,h,j;\,n) = \prod_{m=1}^{n} (3 - xF_{k+m} - yF_{h+m} - zF_{j+m}) = \left(\sum^{(3)}, r\Big|_0^n,\ s\Big|_0^n,\ t\Big|_0^n\right) A(k,h,j;n,n,n;r,s,t)\cdot x^r y^s z^t, \qquad r+s+t \le n .$$

(20)
$$\eta(x,y,z;\,k,h,j;\,0) = 1 .$$

To find an inversion formula for (19) we use (3), i.e.,

$$x^r = \sum_{m=0}^{r} B(k,r,m)\eta(x,k,m)$$

$$y^s = \sum_{p=0}^{s} B(h,s,p)\eta(y,h,p)$$

$$z^t = \sum_{q=0}^{t} B(j,t,q)\eta(z,j,q) ,$$

so that

(21)
$$x^r y^s z^t = \left(\sum{}^{(3)}, m\Big|_0^r, p\Big|_0^s, q\Big|_0^t\right) B(k,r,m)B(h,s,p)B(j,t,q) \cdot \eta(x,k,m)\eta(y,h,p)\eta(z,j,q)$$
$$= \left(\sum{}^{(3)}, m\Big|_0^r, p\Big|_0^s, q\Big|_0^t\right) B(k,h,j;\ r,s,t;\ m,p,q) \cdot \eta(x,k,m)\eta(y,h,p)\eta(z,j,q) \quad ,$$

where

(22)
$$B(k,h,j;\ r,s,t;\ m,p,q) = B(k,r,m)B(h,s,p)B(j,t,q).$$

5. QUASI-ORTHOGONALITY RELATIONS

If in the second form of (21) we substitute according to (1) we obtain

$$x^r y^s z^t = \left(\sum{}^{(3)}, m\Big|_0^r, p\Big|_0^s, q\Big|_0^t\right) B(k,h,j;\ r,s,t;\ m,p,q) \sum_{a=0}^{m} A(k,m,a)x^a \cdot \sum_{b=0}^{p} A(h,p,b)y^b \sum_{c=0}^{q} A(j,q,c)z^c \quad ,$$

$$= \left(\sum{}^{(6)}, m\Big|_0^r, p\Big|_0^s, q\Big|_0^t, a\Big|_0^m, b\Big|_0^p, c\Big|_0^q\right) B(k,h,j;\ r,s,t;\ m,p,q)\, A(k,m,a)A(h,p,b)A(j,q,c)x^a y^b z^c \ .$$

Since the A and B numbers are zero under the conditions stated in the introduction we can extend the limits m, p, q of the summation to n, change the order of summations, and obtain after taking out the zero coefficients

(23)
$$\left(\sum{}^{(3)}, m\Big|_a^r, p\Big|_b^s, q\Big|_c^t\right) B(k,h,j;\ r,s,t;\ m,p,q)A(k,m,a)A(h,p,b) \cdot A(j,q,a) = \delta_a^r \delta_b^s \delta_c^t \ .$$

This relation is actually nothing but the product of three relations of the form given by (6).

6. RECURRENCE RELATIONS

By writing

$$\eta(x, y, z; k, h, j; n + 1) = (3 - xF_{k+n+1} - yF_{h+n+1} - zF_{j+n+1})\eta(x, y, z; k, h, j, n)$$

and substituting according to (19) and equating the coefficients of the same monomials we obtain

$$\begin{aligned} A(k, h, j; n + 1, n + 1, n + 1; r, s, t) &= 3A(k, h, j; n, n, n; r, s, t) \\ &- F_{k+n+1}A(k, h, j; n, n, n; r - 1, s, t) - F_{h+n+1}A(k, h, j; n, n, n; r, s - 1, t) \\ &- F_{j+n+1}A(k, h, j; n, n, n; r, s, t - 1) \quad , \end{aligned} \tag{25}$$

which is a recurrence realtion satisfied by the A numbers.

To find a recurrence relation satisfied by the B numbers we use (8) and obtain

$$\begin{aligned} B(k, r, m) &= (F_{m+1+k})^{-1}B(k, r - 1, m) - (F_{m+k})^{-1}B(k, r - 1, m - 1) \\ B(h, s, p) &= (F_{p+1+h})^{-1}B(h, s - 1, p) - (F_{p+h})^{-1}B(h, s - 1, p - 1) \\ B(j, t, q) &= (F_{q+1+j})^{-1}B(j, t - 1, q) - (F_{q+j})^{-1}B(j, t - 1, q - 1) \ , \end{aligned}$$

and by multiplying these three relations by each other and using (22) we have the following recurrence relation for the B numbers:

$$\begin{aligned} B(k, h, j; r, s, t; m, p, q) &= (F_{m+1+k}F_{p+1+h}F_{q+1+j})^{-1} \cdot \\ &\quad B(k, h, j; r - 1, s - 1, t - 1; m, p, q) \\ &-(F_{m+1+k}F_{p+1+h}F_{q+j})^{-1}B(k, h, j; r - 1, s - 1, t - 1; m, p, q - 1) \\ &-(F_{m+1+k}F_{p+h}F_{q+1+j})^{-1}B(k, h, j; r - 1, s - 1, t - 1; m, p - 1, q) \\ &-(F_{m+k}F_{p+1+h}F_{q+1+j})^{-1}B(k, h, j; r - 1, s - 1, t - 1; m - 1, p, q) \\ &+(F_{m+1+k}F_{p+h}F_{q+j})^{-1}B(k, h, j; r - 1, s - 1, t - 1; m, p - 1, q - 1) \\ &+(F_{m+k}F_{p+1+h}F_{q+j})^{-1}B(k, h, j; r - 1, s - 1, t - 1; m - 1, p, q - 1) \\ &+(F_{m+k}F_{p+h}F_{q+1+j})^{-1}B(k, h, j; r - 1, s - 1, t - 1, m - 1, p - 1, q) \\ &-(F_{m+k}F_{p+h}F_{q+j})^{-1}B(k, h, j; r - 1, s - 1, t - 1; m - 1, p - 1, q - 1) \ . \end{aligned}$$

7. CONCLUDING REMARKS

(i) Equations (7), (8), (12), (13), (18), (25), and (26) indicate that the coefficients A and B involved are particular solutions of corresponding partial difference equations which may be of interest.

(ii) Although in this paper we have assumed that the numbers F_k are Fibonacci numbers the same relations would hold for any sequence that is defined for k being a positive integer or zero.

(iii) We have not attempted to define Lah numbers corresponding to the A and B numbers in the case of several variables although this seems possible.

REFERENCES

1. S. Tauber, "On Quasi-Orthogonal Numbers," Am. Math. Monthly, 69 (1962), pp. 365-372.

2. S. Tauber, "On Generalized Lah Numbers," Proc. Edinburgh Math. Soc., 14 (Series II), (1965), pp. 229-232.

3. H. W. Gould, "The Construction of Orthogonal and Quasi-Orthogonal Number Sets," Am. Math. Monthly, 72 (1965), pp. 591-602.

* * * * *