THE LINEAR DIOPHANTINE EQUATION IN n VARIABLES AND ITS APPLICATION TO GENERALIZED FIBONACCI NUMBERS

LEON BERNSTEIN
Syracuse, New York

1. SUMMARY OF RESULTS

The solution of the Linear Diophantine Equation in n unknowns, viz.

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = c$$

with

$$n \geq 2; \; c_1, c_2, \cdots, c_n, \; c$$

integers is a problem which may occupy more space in the future development of linear programming. For $n = 2$ this is achieved by known methods — either by developing c_2/c_1 in a continued fraction by Euclid's algorithm or by solving the linear congruence $c_1x_1 \equiv c(c_2)$. For $n > 2$ refuge is usually taken to solving separately the equation $c_1x_1 + c_2x_2 = c$ and the homogeneous linear equation $c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0$ and adding the general solution of the latter to a special solution of the former. This is usually a most cumbersome method which becomes especially unhappy under the restriction that none of the unknowns $x_i (i = 3, \cdots, n)$ vanishes, since in the opposite case the rank of the Diophantine equation is lowered. The first part of the present paper, therefore, suggests a method of solving the linear Diophantine equation in $n > 2$ unknowns with the restriction $x_i \neq 0$ $(i = 1, \cdots, n)$ based on a modified algorithm of Jacobi-Perron; it is proved that if the equation is consistent, this method always leads to a solution; numerical examples illustrate the theory.

In the second part of this paper these results are being used to state explicitly the solution of a linear Diophantine equation whose coefficients are generalized Fibonacci numbers. The periodicity of the ratios of generalized Fibonacci numbers of the third degree is proved using rational ratios only.

Concluding, an explicit formula is stated for the limiting ratio of two subsequent generalized Fibonacci numbers of any degree by means of two simple infinite series. For this purpose the author repeatedly utilizes results of his previous papers on a modified algorithm of Jacobi-Perron.

2. THE STANDARD EQUATION

A Linear Diophantine Equation in n unknowns

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = 1, \quad n > 2 \tag{1.1}$$

will be called a Standard Equation of Degree n (abbreviated S.E.n) if the following restrictions on its coefficients hold:

(1.2)

a) c_i a natural number for every $i = 1, \cdots, n$;
b) $1 < c_1 < c_2 < \cdots < c_n$;
c) $(c_1, c_2, \cdots, c_n) = 1$;
d) $c_i \nmid c_{i+j}$; $i, j \geq 1$, $i + j \leq n$;
e) $(c_{k_1}, c_{k_2}, \cdots, c_{k_{n-1}}) = d > 1$; $k_i, k_j = 1, \cdots, n$; $k_i \neq k_j$; $(i, j = 1, \cdots, n-1)$.

A linear Diophantine equation in m unknowns with integral coefficients

$$a_1y_1 + a_2y_2 + \cdots + a_my_m = A, \quad (m > 1;\ a_i \neq 0;\ i = 1, \cdots, m) \tag{1.3}$$

will be called trivial, if

$$a_i = 1 \text{ for at least one } i; \tag{1.4}$$

otherwise it will be called nontrivial. This notation is justified; for let be $|a_i| = 1$ in (1.3). Then all the solutions of (1.3) are given by

(1.5)

$$y_1, y_2, \cdots, y_{i-1}, y_{i+1}, \cdots, y_m \text{ any integers}, \quad 1 < i < m;$$
$$y_i = a_i(A - a_1y_1 - a_2y_2 - \cdots - a_{i-1}y_{i-1} - a_{i+1}y_{i+1} - \cdots - a_my_m);$$

and similar for $i = 1$, $i = m$.

Let equation (1.3) be nontrivial; it will be called reduced, if

$$(1.6) \qquad (a_1, a_2, \cdots, a_m, A) = 1$$

nonreduced, if

$$(1.7) \qquad (a_1, a_2, \cdots, a_m, A) = d > 1 .$$

With the meaning of (1.7), (1.3) can always w. l. o. g. be reduced by cancelling d from the coefficients $a_1, \cdots, a_m, A$.

As is well known, (1.3) is solvable if

$$(1.8) \qquad (a_1, a_2, \cdots, a_m) \mid A ,$$

otherwise unsolvable.

Theorem 1.1. Every reduced nontrivial solvable equation (1.3) can be transformed into an S. E. n.

Proof. We obtain from the conditions of Theorem 1.1.

$$(1.9) \qquad (a_1, a_2, \cdots, a_m, A) = 1; \qquad |a_i| > 1, \quad (i = 1, \cdots, m) .$$

Substituting in (1.3)

$$(1.10) \qquad y_i = Az_i , \quad (i = 1, \cdots, m)$$

we obtain

$$(1.11) \qquad a_1z_1 + a_2z_2 + \cdots + a_mz_m = 1 .$$

Since (1.3) is solvable, we have $(a_1, a_2, \cdots, a_m) \mid A$, which, together with (1.9), yields

$$(1.12) \qquad (a_1, a_2, \cdots, a_m) = 1 .$$

Let denote

$$z_{k_i} = u_{k_i} \quad \text{if} \quad b_{k_i} = a_{k_i} > 0 \,, \tag{1.13}$$

$$z_{k_i} = -u_{k_i} \ \text{if} \ b_{k_i} = -a_{k_i} > 0 \quad (k_i = 1, \cdots, m) \,. \tag{1.14}$$

In virtue of (1.13), (1.14), equation (1.11) takes the form

$$b_1u_1 + b_2u_2 + b_mu_m = 1; \quad (b_1, b_2, \cdots, b_m) = 1 \,. \tag{1.15}$$

We can now presume, without loss of generality,

$$1 < b_1 \le b_2 \le b_3 \le \cdots \le b_m \,. \tag{1.16}$$

Let b_i be the first coefficient in (1.16) such that

$$b_i \mid b_{k_s}, k_s > i, s = 1, \cdots, m-n; \ m-n \quad m-i; \ i+1 \le k_s \le m \,. \tag{1.17}$$

Putting

$$\begin{aligned} b_{k_s} &= t_s b_i \,, \qquad (s = 1, \cdots, m-n) \\ u_i + t_1u_{k_1} + t_2u_{k_2} + \cdots + t_{m-n}\, u_{k_{m-n}} &= v_i \,, \end{aligned} \tag{1.18}$$

we obtain from (1.15), (1.18)

$$\begin{aligned} &b_1u_1 + b_2u_2 + \cdots + b_{i-1}u_{i-1} + b_iv_i + b_{r_1}u_{r_1} + \cdots + b_{r_{n-i}}u_{r_{n-i}} = 1, \\ &b_i \nmid b_{r_1}, b_{r_2}, \cdots, b_{r_{n-i}}; \ i+1 \le r_q \le m, \quad (q = 1, \cdots, n-i) \,. \end{aligned} \tag{1.19}$$

We shall prove

$$(b_1, b_2, \cdots, b_{i-1}, b_i, b_{r_1}, b_{r_2}, \cdots, b_{r_{n-i}}) = 1 \,. \tag{1.20}$$

Suppose,

$$(b_1, b_2, \cdots, b_{i-1}, b_i, b_{r_1}, b_{r_2}, \cdots, b_{r_{n-i}}) = d > 1;$$

we would then obtain, in view of (1.17),

$$(b_1, b_2, \cdots, b_i, b_{i+1}, \cdots, b_m) =$$
$$(b_1, b_2, \cdots, b_{i-1}, b_i, b_{k_1}, \cdots, b_{k_{m-n}}, b_{r_1}, \cdots, b_{r_{n-i}}) \geq$$
$$(b_1, b_2, \cdots, b_{i-1}, b_i, b_{r_1}, b_{r_2}, \cdots, b_{r_{n-i}}) = d > 1 ,$$

contrary to (1.15).

If there exists a b_{r_q} such that $b_{r_q} | b_{r_p}$, $(p > q)$ this process is repeated as before; otherwise we obtain from (1.19) denoting

$$\begin{aligned} &b_j = h_j, \quad (j = 1, \cdots, i); \; u_j = v_j, \quad (j = 1, \cdots, i - 1) ; \\ &b_{r_j} = h_{i+j} ; \; u_{r_j} = v_{i+j} , \quad (j = 1, \cdots, n - i) , \end{aligned} \tag{1.21}$$

$$\begin{aligned} &h_1v_1 + h_2v_2 + \cdots + h_iv_i + h_{i+1}v_{i+1} + \cdots + h_nv_n = 1 , \\ &1 < h_1 < h_2 < \cdots < h_n ; \quad (h_1, \cdots, h_n) = 1, \; h_i \nmid h_j; \; j > i. \end{aligned} \tag{1.22}$$

It should be noted that, in virtue of (1.18), the values of $u_1, u_{k_1}, u_{k_2}, \cdots, u_{k_{m-n}}$ are obtained from those of v_i in (1.22) as follows

$$u_{k_1}, \cdots, u_{k_{m-n}} \text{ any integers; } u_i = v_i - t_1u_{k_1} - \cdots - t_{m-n}u_{k_{m-n}}. \tag{1.23}$$

If the h_i $(i = 1, \cdots, n)$ of (1.22) do not fulfill conditions e) of (1.2), we choose n different primes p_i such that

$$p_i \nmid h_1h_2 \cdots h_n , \quad (i = 1, \cdots, n) ; \; p_1 > p_2 > \cdots > p_n , \tag{1.24}$$

and denote

$$p_1p_2 \cdots p_n = P; \; v_i = p_i^{-1}Px_i ; \; c_i = p_i^{-1}Ph_i , \quad (i = 1, \cdots, n) . \tag{1.25}$$

With (1.25) equation (1.22) takes the form (1.1). Since

$$c_1 = h_1p_1^{-1}P = h_1p_2p_3 \cdots p_m > h_1 ,$$

we obtain

(1.26) $$c_1 > 1$$

We further obtain, for $i \geq 1$, and in virtue of (1.24)

(1.27) $$c_i = h_i p_i^{-1} P < h_{i+1} p_i^{-1} P < h_{i+1} p_{i+1}^{-1} P = c_{i+1} ,$$
$$c_i < c_{i+1} \quad (i = 1, 2, \cdots, n-1) .$$

But

$$(p_1^{-1}P, \cdots, p_n^{-1}P) = 1, \quad \text{and} \quad (h_1, h_2, \cdots, h_n) = 1 ,$$

and since $p_i \nmid h_1 h_2 \cdots h_n$, we obtain, on ground of a known theorem

$$(h_1 p_1^{-1}P, h_2 p_2^{-1}, \cdots, h_n p_n^{-1}P) = 1 ,$$

so that

(1.28) $$(c_1, c_2, \cdots, c_n) = 1 .$$

We shall now prove that the numbers c_i $(i = 1, \cdots, n)$ from (1.25) fulfill the conditions e) of (1.2). We shall prove it for one $(n - 1)$ tuple of the c_i; the general proof for any $(n - 1)$ tuple is analogous. We obtain

$$(c_1, c_2, \cdots, c_{n-1}) = (h_1 p_1^{-1}P, h_2 p_2^{-1}P, \cdots, h_{n-1} p_{n-1}^{-1}P) =$$
$$(h_1 p_2 p_3 \cdots p_n, h_2 p_1 p_3 \cdots p_n, \cdots, h_{n-1} p_1 \cdots p_{n-2} p_n) \geq p_n > 1 .$$

By this method we obtain, indeed, generally

(1.29) $$(c_{k_1}, c_{k_2}, \cdots, x_{k_{n-1}}) = p_{k_n} > 1, \quad k_i \neq k_j \text{ for } i \neq j .$$

Thus Theorem 1.1 is completely proved.

A Linear Diophantine Equation in n unknowns which satisfies conditions a), b), c), d) of (1.1) will be called a Deleted Standard Equation of Degree n (abbreviated S'. E. n). Let

$$h_1v_1 + h_2v_2 + \cdots + h_nv_n = 1$$

be an S'. E. n. We have proved that every nontrivial reduced solvable Diophantine equation can be transformed into an S'. E. n, whereby $n > 2$.

An n-tuple of integers $(x_1, x_2, \cdots, x_n)$ for which

$$h_1x_1 + h_2x_2 + \cdots + h_nx_n = 1 \quad , \tag{1.30}$$

is a solution vector of S'. E. n; it will be called a standard solution vector, if $x_i \neq 0$ for all $i = 1, \cdots, n$. As already pointed out in the Summary of Results, we are aiming at finding a standard solution vector of S'. E. n. Since in the S'. E. n condition e) of (1.2) it is not fulfilled, there must be at least one (n - 1)-tuple of numbers among the $h_1, \cdots, h_n$ which are relatively prime. We shall presume, without loss of generality,

$$(h_1, h_2, \cdots, h_{n-1}) = 1 \tag{1.31}$$

and let $(x_1, x_2, \cdots, x_{n-1})$ be a standard solution vector of

$$h_1v_1 + h_2v_2 + \cdots + h_{n-1}v_{n-1} = 1 \quad .$$

Then $(x_1, x_2, \cdots, x_{n-1}, 0)$ is a solution vector of the S'. E. n, but it is not a standard solution vector; such one would be given by the n-tuple,

$$(x_1, x_2, \cdots, x_{n-1} - th_n, th_{n-1}),$$
$$t \text{ any integer}, \quad x_{n-1} \neq th_n .$$

Thus the problem for an S'. E. n which is not an S. E. n is reduced to find a standard solution vector of an S'. E. n - 1; this can be either an S . E. n - 1, or only an S'. E. n - 1.

<u>Theorem 1.2.</u> An S. E. n has only standard solution vectors.

<u>Proof.</u> Let $(x_1, x_2, \cdots, x_k, 0, 0, \cdots, 0)$ be a solution vector of an S. E. n, and let $x_i \neq 0$, $(i = 1, \cdots, k)$. It is easy to verify that $k \geq 2$, and let be $k \leq n - 1$. The arrangement of the components of the solution vector can be assumed without loss of generality. Then

(1.32) $$c_1x_1 + c_2x_2 + \cdots + c_kx_k = 1 ;$$

but since

$$(c_1, c_2, \cdots, c_k, c_{k+1}, \cdots, c_{n-1}) = p_n ,$$

we obtain

$$(c_1, c_2, \cdots, c_k) \geq p_n > 1 ,$$

which is inconsistent with (1.32). This proves Theorem 1.2.

Let again

$$h_1v_1 + h_2v_2 + \cdots + h_nv_n = 1$$

be an S'. E. n and

(1.33) $$h_1v_1 + h_2v_2 + \cdots + h_nv_n = 0$$

its homogeneous part. We shall denote

(1.34) $$D(h_1, \cdots, h_n) = \begin{vmatrix} th_1 & v_{1,1} & v_{1,2} & \cdots & v_{1,n-2} & h_1 \\ th_2 & v_{2,1} & v_{2,2} & \cdots & v_{2,n-2} & h_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ th_n & v_{n,1} & v_{n,2} & \cdots & v_{n,n-2} & h_n \end{vmatrix}$$

t, $v_{i,j}$ any integers,
$(i = 1, \cdots, n;\ j = 1, \cdots, n-2)$

(1.35) $H_{k,n}$ is the algebraic cofactor of the element $a_{k,n}$.

For any $v_{i,j}$ the following identity holds

(1.36) $$D(h_1, \cdots, h_n) = h_1H_{1,n} + h_2H_{2,n} + \cdots + h_nH_{n,n} = 0.$$

Theorem 1.3. Let $(x_1, x_2, \cdots, x_n)$ be a solution vector of an S'. E. n and $(H_{1,n}, H_{2,n}, \cdots, H_{n,n})$ be any solution vector of its homogeneous part; then infinitely many solution vectors of S'. E. n are given by

$$(1.37) \qquad (x_1 + H_{1,n}, x_2 + H_{2,n}, \cdots, x_n + H_{n,n}) \ .$$

Proof. This follows immediately from (1.30), (1.36) adding these two equations.

2. A MODIFIED ALGORITHM OF JACOBI-PERRON

Pursuing ideas of Jacobi [2] and Perron [3], the author [1, a) - q)] has modified the algorithm named after the two great mathematicians (see especially [1, m), n), p)]; one of these [1, p)] will be used in the second part of this paper. In order to find a standard solution vector of an S'. E. n, the author suggests a new modification of the Jacobi-Perron algorithm as outlined below.

We shall denote, as usually, by V_{n-1} the set of all ordered (n - 1)-tuples of real numbers $(a_1, a_2, \cdots, a_{n-1})$, $(n = 2, 3, \cdots)$ and call V_{n-1} the real number vector space of dimension n - 1 and the (n - 1)-tuples its vectors. Let

$$(2.1) \qquad a^{(0)} = (a_1^{(0)}, a_2^{(0)}, \cdots, a_{n-1}^{(0)})$$

be a given vector in V_{n-1}, and let

$$(2.2) \qquad b^{(v)} = (b_1^{(v)}, b_2^{(v)}, \cdots, b_{n-1}^{(v)})$$

be a sequence of vectors in V_{n-1}, which are either arbitrarily given or derived from $a^{(0)}$ by a certain transformation of V_{n-1}. We shall now introduce the following transformation

$$(2.3) \qquad Ta^{(v)} = a^{(v+1)} = \frac{1}{a_1^{(v)} - b_1^{(v)}} (a_2^{(v)} - b_2^{(v)}, \cdots, a_{n-1}^{(v)} - b_{n-1}^{(v)}, 1)$$

$$a_1^{(v)} \neq b_1^{(v)}, \quad v = 0, 1, \cdots$$

If we define the real numbers $A_i^{(v)}$ by the recursion formulas

$$A_i^{(i)} = 1; \quad A_i^{(v)} = 0; \quad (i,\ v = 0, 1, \cdots, n-1;\ i \neq v)\ ,$$

$$A_i^{(v+n)} = A_i^{(v)} + \sum_{j=1}^{n-1} b_j^{(v)} A_i^{(v+j)}, \quad (i = 0, \cdots, n-1;\ v = 0, 1, \cdots)$$

then, as has been proved by the author and previously stated by Perron, the following formulas hold

$$(2.5) \qquad D_v = \begin{vmatrix} A_0^{(v)} & A_0^{(v+1)} & \cdots & A_0^{(v+n-1)} \\ A_1^{(v)} & A_1^{(v+1)} & \cdots & A_1^{(v+n-1)} \\ \cdot & \cdot & \cdot & \cdot \\ A_{n-1}^{(v)} & A_{n-1}^{(v+1)} & \cdots & A_{n-1}^{(v+n-1)} \end{vmatrix} = (-1)^{v(n-1)}, \quad (v = 0, 1, \cdots)$$

$$(2.6) \qquad a_i^{(0)} = \frac{A_i^{(v)} + \sum_{j=1}^{n-1} a_j^{(v)} A_i^{(v+j)}}{A_0^{(v)} + \sum_{j=1}^{n-1} a_j^{(v)} A_0^{(v+j)}}, \quad (i = 1, \cdots, n-1;\ v = 0, 1, \cdots)$$

(2.5) is the determinant of the transformation matrix of $Ta^{(v)}$; a further important formula proved by the author in [1, p] is

$$(2.6a) \qquad \begin{vmatrix} 1 & A_0^{(v+1)} & \cdots & A_0^{(v+n-1)} \\ a_1^{(0)} & A_1^{(v+1)} & \cdots & A_1^{(v+n-1)} \\ a_2^{(0)} & A_2^{(v+1)} & \cdots & A_2^{(v+n-1)} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n-1}^{(0)} & A_{n-1}^{(v+1)} & \cdots & A_{n-1}^{(v+n-1)} \end{vmatrix} = \frac{(-1)^{v(n-1)}}{A_0^{(v)} + \sum_{j=1}^{n-1} a_j^{(v)} A_0^{(v+j)}}$$

$v = 0, 1, \cdots$

In the previous papers of the author the vectors $b^{(v)}$ were not arbitrarily chosen, but derived from the vectors $a^{(v)}$ by a special formation law. The nature of this formation law plays a decisive role in the theory of the modified algorithms of Jacobi-Perron. Both Jacobi and my admired teacher Perron used only the formation law:

$$b_i^{(v)} = [a_i^{(v)}], \quad (i = 1, \cdots, n-1; \; v = 0, 1, \cdots) \tag{2.7}$$

where $[x]$ denotes, as customary, the greatest integer not exceeding x. In this paper the modification of Jacobi-Perron's algorithm rests with the following different formation law of the $b_i^{(v)}$

$$\begin{aligned} b_1^{(v)} &= a_1^{(v)} \quad \text{if } a_1^{(v)} \neq [a_1^{(v)}]; \\ b_1^{(v)} &= a_1^{(v)} - 1 \quad \text{if } a_1^{(v)} = [a_1^{(v)}]; \\ b_k^{(v)} &= [a_k^{(v)}] \qquad (k = 2, \cdots, n-1; \; v = 0, 1, \cdots). \end{aligned} \tag{2.8}$$

It may happen that for some v $a_i^{(v)} = [a_i^{(v)}]$ for every i. In this case the algorithm with the formation law (2.8) must be regarded as finished, and $b_i^{(v)} = a_i^{(v)}$, $(i = 1, \cdots, n-1)$. The algorithm of the vectors $a^{(v)}$ as given by (2.3) is called periodic if there exist two integers p, q $(p \geq 0, \; q \geq 1)$ such that the transformation T yields

$$T^{v+q} = T^v, \quad (v = p, p+1, \cdots) \tag{2.9}$$

In case of periodicity the vectors $a^{(v)}$ $(v = 0, p, \cdots, p-1)$ are said to form the preperiod, and the vectors $a^{(v)}$ $(v = p, p+1, \cdots, p+q-1)$ are said to form the period of the algorithm; min p = s and min q = t are called respectively the lengths of the preperiod and period; s + t is called the length of the algorithm which is purely periodic if s = 0.

3. A STANDARD SOLUTION VECTOR OF S. E. n

Let

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = 1 \tag{3.1}$$

be an S. E. n; let the given vector $a^{(0)}$ in V_{n-1} have the form

$$a^{(0)} = (a_1^{(0)}, \cdots, a_{n-1}^{(0)}); \quad a_i^{(0)} = c_{i+1}/c_1 \qquad (i = 1, \cdots, n-1). \tag{3.2}$$

The main result of this chapter is stated in

Theorem 3.1. Let the vectors $a^{(v)}$ be transforms of the vector $a^{(0)}$ from (3.2), obtained from (2.3) by means of the formation law (2.8); then there exists a natural number t such that the components of the vector $a^{(t)}$ are integers, viz.

$$a^{(t)} = (a_1^{(t)}, \cdots, a_{n-1}^{(t)}), \qquad a_i^{(t)} \text{ integers} \quad (i = 1, \cdots, n-1). \tag{3.3}$$

Proof. We obtain from (2.8), since $c_1 \nmid c_2$ and, therefore, $[a_1^{(0)}] \neq a_1^{(0)}$,

$$b_i^{(0)} = [c_{i+1}/c_1], \qquad (i = 1, \cdots, n-1). \tag{3.4}$$

From (3.4) we obtain

$$\begin{aligned} &c_{i+1} = b_i^{(0)}c_1 + c_i^{(1)}, \qquad (c_i^{(1)} \text{ an integer}), \\ &0 < c_i^{(1)} < c_n^{(1)}; \quad c_n^{(1)} = c_1; \quad (i = 1, \cdots, n-1) \end{aligned} \tag{3.5}$$

From (3.2), (3.4) and (3.5) we obtain

$$a_i^{(0)} - b_i^{(0)} = \frac{c_{i+1}}{c_1} - \frac{c_{i+1} - c_i^{(1)}}{c_1},$$

$$a_1^{(0)} - b_1^{(0)} = \frac{c_1^{(1)}}{c_1}; \quad a_{i+1}^{(0)} - b_{k+1}^{(0)} = \frac{c_{k+1}^{(1)}}{c_1}, \quad (k = 1, \cdots, n-2) \tag{3.6}$$

and from (3.6), in view of (2.3)

$$(3.7)\qquad a_i^{(1)} = c_{i+1}^{(1)} \Big/ c_1^{(1)} , \quad (i = 1,\cdots,n-1) ,$$

so that

$$(3.8)\qquad \begin{aligned} b_i^{(1)} &= [c_{i+1}^{(1)} / c_1^{(1)}] , && (i = 2, \cdots, n-1) ; \\ b_1^{(1)} &= [c_2^{(1)} / c_1^{(1)}] , && \text{if } c_1^{(1)} \nmid c_2^{(1)} , \\ b_1^{(1)} &= (c_2^{(1)} / c_1^{(1)}) - 1, && \text{if } c_1^{(1)} \mid c_2^{(1)} . \end{aligned}$$

If $c_1^{(1)} = 1$, Theorem 3.1 is true with $t = 1$; let us, therefore, presume that $c_1^{(1)} > 1$. Of the two possible cases, viz. I) $c_1^{(1)} \mid c_2^{(1)}$ and II) $c_1^{(1)} \nmid c_2^{(1)}$, we shall first investigate case II). Here we obtain

$$(3.9)\qquad \begin{aligned} & c_{i+1}^{(1)} = b_i^{(1)} c_1^{(1)} + c_i^{(2)} , \qquad (c_i^{(2)} \text{ an integer}) , \\ & 0 \le c_i^{(2)} < c_n^{(2)} ; \quad c_n^{(2)} = c_1^{(1)} , \quad (i = 2,\cdots,n-1) ; \\ & 0 < c_1^{(2)} < c_n^{(2)} \end{aligned}$$

We obtain, comparing (3.5) and (3.9)

$$(3.10)\qquad 0 < c_1^{(2)} < c_1^{(1)} < c_1 .$$

Before investigating case I), we shall prove the following

Lemma 3.1.1. Let the vector $a^{(v)}$ in the modified algorithm of Jacobi-Perron with the formation law (2.8) and the given vector (3.2) have the form

$$(3.11)\qquad a^{(v)} = \left(\frac{c_2^{(v)}}{c_1^{(v)}} , \frac{c_3^{(v)}}{c_1^{(v)}} , \cdots , \frac{c_n^{(v)}}{c_1^{(v)}} \right) \quad (v = 0, 1, \cdots)$$

then

$$(3.12) \qquad (c_1^{(v)}, c_2^{(v)}, \cdots, c_n^{(v)}) = 1 .$$

Proof. The lemma is correct for $v = 0$, in virtue of (3.1) and (3.2). Let it be true for $v = k$, viz.

$$(3.13) \qquad a^{(k)} = \frac{1}{c_1^{(k)}} (c_2^{(k)}, c_3^{(k)}, \cdots, c_n^{(k)}), \ (c_1^{(k)}, c_2^{(k)}, \cdots, c_n^{(k)}) = 1 .$$

From (3.13) we obtain

$$(3.14) \qquad \begin{gathered} c_{i+1}^{(k)} = b_i^{(k)} c_1^{(k)} + c_i^{(k+1)} ; \ c_i^{(k+1)} \text{ integers}, \quad (i = 1, \cdots, n-1). \\ 0 < c_i^{(k+1)} < c_1^{(k)} \quad . \end{gathered}$$

Let us denote

$$(3.15) \qquad c_1^{(k)} = c_n^{(k+1)} ,$$

$$(3.16) \qquad (c_1^{(k+1)}, c_2^{(k+1)}, \cdots, c_n^{(k+1)}) = d .$$

If $d = 1$, Lemma 3.1.1 is proved; let us, therefore, presume

$$(3.17) \qquad d > 1 .$$

We then obtain from (3.14), (3.15), (3.16)

$$(3.18) \qquad d \mid c_n^{(k+1)}; \quad c_n^{(k+1)} = c_1^{(k)} ; \quad d \mid c_{i+1}^{(k)} , \qquad (i = 1, \cdots, n-1) ,$$

so that

$$(3.19) \qquad (c_1^{(k)}, c_2^{(k)}, \cdots, c_n^{(k)}) \geq d > 1 ;$$

but (3.19) contradicts (3.13), and the assumption that $d > 1$ is false which proves the lemma. We shall return to case I) and presume

$$c_1^{(1)} \mid c_{i+1}^{(1)} \,, \quad (i = 1, 2, \cdots, m) \,. \tag{3.20}$$

In view of Lemma 3.1.1, the restriction holds

$$m \leq n - 2 \,, \tag{3.21}$$

since, permitting $m = n - 1$, we would obtain

$$(c_1^{(1)}, \cdots, c_n^{(1)}) = c_1^{(1)} > 1 \,,$$

contrary to Lemma 3.1.1. It then follows from (3.20), in view of (2.8)

$$\begin{aligned} & c_2^{(1)} = (b_1^{(1)} + 1)c_1^{(1)} \,;\ c_{i+1}^{(1)} = b_i^{(1)} c_1^{(1)} \,, \quad (i = \ \,, \cdots, m) \,; \\ & c_{m+2}^{(1)} = b_{m+1}^{(1)} c_1^{(1)} + c_{m+1}^{(2)} \,;\ 1 \leq c_{m+1}^{(2)} \leq c_1^{(1)} \,; \\ & c_{m+2+j}^{(1)} = b_{m+1+j}^{(1)} c_1^{(1)} + c_{m+1+j}^{(2)} \,; \\ & 0 \leq c_{m+1+j}^{(2)} < c_1^{(1)} \,, \qquad (j = 1, \cdots, n - m - 2) \,. \end{aligned} \tag{3.22}$$

From (3.7), (3.22), we obtain, denoting

$$c_1^{(1)} = c_n^{(2)} \tag{3.23}$$

$$\begin{aligned} & a_1^{(1)} - b_1^{(1)} = 1 \,;\ a_i^{(1)} - b_i^{(1)} = 0 \,, \qquad (i = 2, \cdots, m) \,; \\ & a_{m+1}^{(1)} - b_{m+1}^{(1)} = c_{m+1}^{(2)} \Big/ c_n^{(2)} \,; \\ & a_{m+1+j}^{(1)} - b_{m+1+j}^{(1)} = c_{m+1+j}^{(2)} \Big/ c_n^{(2)} \,, \quad (j = 1, \cdots, n - m - 2) \,. \end{aligned} \tag{3.24}$$

From (3.24) we obtain, in view of (2.3),

$$\begin{aligned} & a_i^{(2)} = 0, \quad (i = 1, \cdots, m - 1); \ a_m^{(2)} = c_{m+1}^{(2)} \Big/ c_n^{(2)} \,; \\ & a_{m+j}^{(2)} = c_{m+1+j}^{(2)} \Big/ c_n^{(2)} \,, \qquad (j = 1, \cdots, n - m - 2) \,;\ a_{n-1}^{(2)} = 1 \,. \end{aligned} \tag{3.25}$$

The reader should well note that all the $a_i^{(2)}$ $(i = 1,\cdots,n-1)$ have the same denominator $c_n^{(2)}$; for if $a_i^{(2)} = 0$ we put $a_i^{(2)} = 0/c_n^{(2)}$; if $a_{n-1}^{(2)} = 1$, we put

$$a_{n-1}^{(2)} = c_n^{(2)} \Big/ c_n^{(2)}$$

Combining (3.5) and (3.22), we obtain

$$1 < c_{m+1}^{(2)} < c_1^{(1)} < c_1 \; . \tag{3.26}$$

From (3.25) we obtain, in view of (2.8) and recalling that

$$c_{m+1+j}^{(2)} < c_1^{(1)} = c_n^{(2)}, \quad (j = 1,\cdots,n-m-2) \; ,$$

$$b_1^{(2)} = -1; \; b_{i+1}^{(2)} = 0 \; ; \; (i = 1,\cdots,n-3) \; b_{n-1}^{(2)} = 1 \; , \tag{3.26a}$$

and from (3.25), (3.26a)

$$\begin{aligned} &a_1^{(2)} - b_1^{(2)} = 1; \; a_{1+i}^{(2)} - b_{1+i}^{(2)} = 0 \; , \quad (i = 1,\cdots,m-2) \; ; \\ &a_m^{(2)} - b_m^{(2)} = c_{m+1}^{(2)} \Big/ c_n^{(2)} \\ &a_{m+j}^{(2)} - b_{m+j}^{(2)} = c_{m+1+j}^{(2)} \Big/ c_n^{(2)} \; , \quad (j = 1,\cdots,n-m-2) \; ; \\ &a_{n-1}^{(2)} - b_{n-1}^{(2)} = 0 \; . \end{aligned} \tag{3.27}$$

From (3.27), we obtain, in view of (2.3),

$$\begin{aligned} &a_i^{(3)} = 0 \; , \quad (i = 1,\cdots,m-2); \; a_{m-1}^{(3)} = c_{m+1}^{(2)} \Big/ c_n^{(2)} \; ; \\ &a_{m-1+j}^{(3)} = c_{m+1+j}^{(2)} \Big/ c_n^{(2)} \; , \quad (j = 1,\cdots,n-m-2) \; ; \\ &a_{n-2}^{(3)} = 0 \; ; \; a_{n-1}^{(3)} = 1 \end{aligned} \tag{3.28}$$

We shall now prove the formula

$$
(3.29)\qquad
\begin{aligned}
&a_i^{(k+1)} = 0, \quad (i = 1,\cdots, m-k); \quad a_{m-k+1}^{(k+1)} = c_{m+1}^{(2)} \Big/ c_n^{(2)} \;; \\
&a_{m-k+1+j}^{(k+1)} = c_{m+1+j}^{(2)} \Big/ c_n^{(2)}, \quad (j = 1,\cdots, n-m-2) \;; \\
&a_{n-k-1+s}^{(k+1)} = 0, \quad (s = 1,\cdots, k-1); \quad a_{n-1}^{(k+1)} = 1 \;; \\
&k = 2,\cdots, m-1 \;.
\end{aligned}
$$

Proof by induction. Formula (3.29) is valid for $k = 2$, in virtue of (3.28). Let it be true for $k = v$, viz.

$$
(3.30)\qquad
\begin{aligned}
&a_i^{(v+1} = 0, \quad (i = 1,\cdots, m-v); \; a_{m-v+1}^{(k+1)} = c_{m+1}^{(2)} \Big/ c_n^{(2)} \\
&a_{m-v+1+j}^{(k+1)} = c_{m+1+j}^{(2)} \Big/ c_n^{(2)}, \quad (j = 1,\cdots, n-m-2) \;; \\
&a_{n-v-1+s}^{(v+1)} = 0, \quad (s = 1,\cdots, v-1) \;; \; a_{n-1}^{(v+1)} = 1 \;.
\end{aligned}
$$

From (3.30) we obtain, in virtue of (2.8) and (3.22),

$$
(3.31)\qquad b_1^{(v+1)} = -1; \; b_{1+i}^{(v+1)} = 0, \quad (i = 1,\cdots, n-3); \; b_{n-1}^{(v+1)} = 1 \;,
$$

and from (3.30) and (3.31),

$$
(3.32)\qquad
\begin{aligned}
&a_1^{(v+1)} - b_1^{(v+1)} = 1 \;; \; a_{1+i}^{(v+1)} - b_{1+i}^{(v+1)} = 0, \quad (i = 1,\cdots, m-v-1); \\
&a_{m-v+1}^{(v+1)} - b_{m-v+1}^{(v+1)} = c_{m+1}^{(2)} \Big/ c_n^{(2)} \;; \\
&a_{m-v+1+j}^{(v+1)} - b_{m-v+1+j}^{(v+1)} = c_{m+1+j}^{(2)} \Big/ c_n^{(2)}, \quad (j = 1,\cdots, n-m-2) \;; \\
&a_{n-v-1+s}^{(v+1)} - b_{n-v-1+s}^{(v+1)} = 0, \quad (s = 1,\cdots, v-1) \\
&a_{n-1}^{(v+1)} - b_{n-1}^{(v+1)} = 0 \;.
\end{aligned}
$$

From (3.32) we obtain, in view of (2.3),

$$a_i^{(v+2)} = 0, \quad (i = 1,\cdots,m-v-1);\ a_{m-v}^{(v+2)} = c_{m+1}^{(2)}\Big/ c_n^{(2)} \ ;$$

(3.33) $$a_{m-v+j}^{(v+2)} = c_{m+1+j}^{(2)}\Big/ c_n^{(2)}, \quad (j = 1,\cdots, n-m-2) \ ;$$

$$a_{n-v-2+s}^{(v+2)} = 0, \quad (s = 1,\cdots,v) \ ; \quad a_{n-1}^{(v+2)} = 1 \ .$$

But (3.33) is formula (3.29) for $k = v + 1$; thus formula (3.29) is completely proved. We now obtain from (3.29), for $k = m - 1$,

$$a_1^{(m)} = 0 \ ; \ a_2^{(m)} = c_{m+1}^{(2)}\Big/ c_n^{(2)} \ ;$$

(3.34) $$a_{2+j}^{(m)} = c_{m+1+j}^{(2)}\Big/ c_n^{(2)}, \quad (j = 1,\cdots,n-m-2) \ ;$$

$$a_{n-m+s}^{(m)} = 0, \quad (s = 1,\cdots,m-2) \ ; \ a_{n-1}^{(m)} = 1 \ ,$$

and from (3.34), in virtue of (2.8) and (3.22)

(3.35) $$b_1^{(m)} = -1; \ b_{1+i}^{(m)} = 0, \quad (i = 1,\cdots,n-3); \ b_{n-1}^{(m)} = 1 \ .$$

From (3.34), (3.35) we obtain

$$a_1^{(m)} - b_1^{(m)} = 1; \ a_2^{(m)} - b_2^{(m)} = c_{m+1}^{(2)}\Big/ c_n^{(2)}$$

(3.36) $$a_{2+j}^{(m)} - b_{2+j}^{(m)} = c_{m+1+j}^{(2)}\Big/ c_n^{(2)}, \quad (j = 1,\cdots,n-m-2) \ ;$$

$$a_{n-m+s}^{(m)} - b_{n-m+s}^{(m)} = 0 \ , \quad (s = 1,\cdots,m-1) \ ,$$

and from (3.36), in view of (2.3)

(3.37) $$a_1^{(m+1)} = c_{m+1}^{(2)}\Big/ c_n^{(2)} \ ; \ a_{1+j}^{(m+1)} = c_{m+1+j}^{(2)}\Big/ c_n^{(2)}, \ (j = 1,\cdots,n-m-2);$$

$$a_{n-m-1+s}^{(m+1)} = 0 \ , \quad (s = 1,\cdots,m-1) \ ; \ a_{n-1}^{(m+1)} = 1 \ .$$

From (3.37) we obtain, in virtue of (2.8) and (3.22),

$$b_i^{(m+1)} = 0, \quad (i = 1,\cdots,n-2)\ ; \ b_{n-1}^{(m+1)} = 1 \quad , \tag{3.38}$$

and from (3.37), (3.38)

$$\begin{aligned} a_1^{(m+1)} - b_1^{(m+1)} &= c_{m+1+j}^{(2)} \Big/ c_n^{(2)} \quad ; \\ a_{1+j}^{(m+1)} - b_{1+j}^{(m+1)} &= c_{m+1+j}^{(2)} \Big/ c_n^{(2)}, \quad (j = 1,\cdots,n-m-2) \\ a_{n-m-1+s}^{(m+1)} - b_{n-m-1+s}^{(m+1)} &= 0 \quad (s = 1,\cdots,m) \end{aligned} \tag{3.39}$$

From (3.39) we obtain, in virtue of (2.3)

$$a_j^{(m+2)} = c_{m+1+j}^{(2)} \Big/ c_{m+1}^{(2)}, \quad (j = 1,\cdots,n-m-2)\ ;$$

$$a_{n-m-2+s}^{(m+2)} = 0, \quad (s = 1,\cdots,m)\ ; \ a_{n-1}^{(m+2)} = c_n^{(2)} \Big/ c_{m+1}^{(2)} \quad ,$$

or

$$\begin{aligned} a_j^{(m+2)} &= c_{j+1}^{(m+2)} \Big/ c_1^{(m+2)}, \quad (j = 1,\cdots,n-m-2)\ ; \\ a_{n-m-2+s}^{(m+2)} &= 0, \quad (s = 1,\cdots,m)\ ; \ a_{n-1}^{(m+2)} = c_n^{(m+2)} \Big/ c_1^{(m+2)}\ ; \\ c_{m+i}^{(2)} &= c_i^{(m+2)}, \quad (i = 1,\cdots,n-m-1)\ ; \ c_n^{(2)} = c_n^{(m+2)} \ . \end{aligned} \tag{3.40}$$

From (3.7), (3.9), we obtain

$$\begin{aligned} a_1^{(1)} - b_1^{(1)} = c_1^{(2)} \Big/ c_1^{(1)}; \ a_{1+j}^{(1)} - b_{1+j}^{(1)} = c_{1+j}^{(2)} \Big/ c_1^{(1)} \quad , \\ (j = 1,\cdots,n-2) \end{aligned} \tag{3.41}$$

and from (3.41), in view of (2.3),

$$(3.42) \qquad a_j^{(2)} = c_{1+j}^{(2)} / c_1^{(2)}, \quad (j = 1, \cdots, n-1); \quad c_1^{(1)} = c_n^{(2)}$$

We have thus obtained two chains of inequalities

$$0 < c_1^{(2)} < c_1^{(1)} < c_1; \quad 0 < c_1^{(m+2)} < c_1^{(1)} < c_1 .$$

If $c_1^{(2)}$ or $c_1^{(m+2)} = 1$, Theorem 3.1 is proved. Otherwise we deduce from (3.40) or (3.42), which show that the vectors $a^{(2)}$ and $a^{(m+2)}$ have the same structure of their components, how the algorithm is to be continued. In any case we obtain a chain of inequalities

$$(3.43) \qquad \begin{gathered} 0 < c_1^{(m_k)} < c_1^{(m_{k-1})} < \cdots < c_1^{(m_2)} < c_1^{(1)} < c_1 , \\ m_2 = 2 \text{ if } c_1^{(1)} \nmid c_2^{(1)}; \; m_2 = m+2 \text{ if } c_1^{(1)} \mid c_2^{(1)}, \cdots \end{gathered}$$

and since the $c_i^{(m_i)}$ are natural numbers, we must necessarily arrive at

$$(3.44) \qquad c_1^{(t)} = 1, \quad t = m_k \geq 1 .$$

This proves Theorem 3.1.

We are now able to state explicitly the standard solution vector of the S. E. n (3.1) and prove, to this end,

<u>Theorem 3.2.</u> A solution vector of the S. E. n is given by the formula

$$(3.45) \qquad \begin{gathered} X = (x_1, x_2, \cdots, x_n); \quad x_i = (-1)^{(t+1)(n-1)} B_{i,n} , \\ (i = 1, \cdots, n) \end{gathered}$$

where the $B_{i,n}$ are the cofactors of the elements of the n^{th} row in the determinant

$$(3.46)\qquad D_{t+1} = \begin{vmatrix} A_0^{(t+1)} & A_0^{(t+2)} & \cdots & A_0^{(t+n)} \\ A_1^{(t+1)} & A_1^{(t+2)} & \cdots & A_1^{(t+n)} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n-1}^{(t+1)} & A_{n-1}^{(t+2)} & \cdots & A_{n-1}^{(t+n)} \end{vmatrix}$$

In D_{t+1} t has the meaning of (3.44) and the

$$A_i^{(v)} \qquad (i = 0, 1, \cdots, n-1;\ v = t+1, t+2, \cdots, t+n)$$

have the meaning of (2.4) and are obtainable from the modified Jacobi-Perron algorithm of the given vector $a^{(0)}$ from (3.2) by means of the formation law (3.8).

Proof. We shall recall that, in virtue of the formation law (3.8) all the numbers $b_i^{(v)}$ and, therefore, the numbers

$$A_i^{(v)} \qquad (i = 0, 1, \cdots, n-1;\ v = 0, 1, \cdots)$$

are integers. For $c_1^{(t)} = 1$ we obtain

$$(3.47)\qquad \begin{aligned} a^{(t)} &= (c_2^{(t)}, c_3^{(t)}, \cdots, c_n^{(t)}) = (a_1^{(t)}, \cdots, a_{n-1}^{(t)}), \\ b_i^{(t)} &= a_i^{(t)} = c_{i+1}^{(t)}, \qquad (i = 1, \cdots, n-1). \end{aligned}$$

Recalling formulas (2.4), (2.6), and (3.2), we obtain

$$a_i^{(0)} = \frac{A_i^{(t)} + \sum_{j=1}^{n-1} a_j^{(t)} A_i^{(t+j)}}{A_0^{(t)} + \sum_{j=1}^{n-1} a_j^{(t)} A_0^{(t+j)}} =$$

$$\frac{A_i^{(t)} + \sum_{j=1}^{n-1} b_j^{(t)} A_i^{(t+j)}}{A_0^{(t)} + \sum_{j=1}^{n-1} b_j^{(t)} A_0^{(t+j)}} = \frac{A_i^{(t+n)}}{A_0^{(t+n)}},$$

so that

$$c_{i+1}/c_1 = A_i^{(t+n)}/A_0^{(t+n)}, \quad (i = 1,\cdots,n-1). \tag{3.48}$$

From (3.48) we obtain

$$c_{i+1} = c_1A_i^{(t+n)}/A_0^{(t+n)},$$

and, since $(c_1, c_2, \cdots, c_n) = 1$,

$$(c_1, c_1A_1^{(t+n)}/A_0^{(t+n)},\ c_1A_2^{(t+n)}/A_0^{(t+n)}, \cdots, c_1A_{n-1}^{(t+n)}/A_0^{(t+n)}) = 1 \tag{3.49}$$

and from (3.49), in virtue of a known theorem,

$$(c_1A_0^{(t+n)}, c_1A_1^{(t+n)}, c_1A_2^{(t+n)}, \cdots, c_1A_{n-1}^{(t+n)}) = A_0^{(t+n)},$$

or

$$c_1(A_0^{(t+n)},\ A_1^{(t+n)},\ A_2^{(t+n)}, \cdots, A_{n-1}^{(t+n)}) = A_0^{(t+n)} \tag{3.50}$$

From (2.5) we obtain

$$D_{t+1} = \begin{vmatrix} A_0^{(t+1)} & A_0^{(t+2)} & \cdots & A_0^{(t+n)} \\ A_1^{(t+1)} & A_1^{(t+2)} & \cdots & A_1^{(t+n)} \\ \cdot & \cdot & \cdots & \cdot \\ A_{n-1}^{(t+1)} & A_{n-1}^{(t+2)} & \cdots & A_{n-1}^{(t+n)} \end{vmatrix} = (-1)^{(t+1)(n-1)}$$

so that

$$(A_0^{(t+n)},\ A_1^{(t+n)},\ A_2^{(t+n)}, \cdots, A_{n-1}^{(t+n)}) = 1. \tag{3.51}$$

From (3.50), (3.51), we obtain

(3.52) $$c_1 = A_0^{(t+n)} ,$$

and from (3.48), (3.52),

(3.53) $$c_{i+1} = A_i^{(t+n)} , \quad (i = 0, 1, \cdots, n-1) .$$

(3.53) is a most decisive result; we obtain, in virtue of it,

(3.54) $$D_{t+1} = \begin{vmatrix} A_0^{(t+1)} & A_0^{(t+2)} & \cdots & A_0^{(t+n-1)} & c_1 \\ A_1^{(t+1)} & A_1^{(t+2)} & \cdots & A_1^{(t+n-1)} & c_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ A_{n-1}^{(t+1)} & A_{n-1}^{(t+2)} & \cdots & A_{n-1}^{(t+n-1)} & c_n \end{vmatrix} = (-1)^{(t+1)(n-1)} ,$$

and from (3.54), denoting the cofactors of the c_i in D_{t+1} by $B_{i,n}$ $(i = 1, \cdots, n)$

$$\sum_{i=1}^{n} B_{i,n} c_i = (-1)^{(t+1)(n-1)} ,$$

or, multiplying both sides of this equation by $(-1)^{(t+1)(n-1)}$,

(3.55) $$\sum_{i=1}^{n} ((-1)^{(t+1)(n-1)} B_{i,n}) c_i = 1 ,$$

which proves Theorem 3.2.

4. NUMERICAL EXAMPLES FOR SOLUTION OF S'. E. n and S. E. n

In this chapter we shall illustrate our theory with three numerical examples.

Let the S'. E. 4 have the form

$$53x + 117y + 209z + 300u = 1 \quad . \tag{4.1}$$

The given vector $a^{(0)}$ has the components

$$a_1^{(0)} = 117/53 \;;\quad a_2^{(0)} = 209/53;\quad a_3^{(0)} = 300/53 \;. \tag{4.2}$$

Carrying out the modified Jacobi-Perron algorithm (2.8) for the vector (4.2), we obtain the sequence of vectors

$$\begin{aligned} b_1^{(0)} &= (2,\ 3,\ 5)\ ; \\ b^{(1)} &= (4,\ 3,\ 4)\ ; \\ b^{(2)} &= (0,\ 1,\ 1)\ ; \\ b^{(3)} &= (1,\ 2,\ 3)\ ; \\ b^{(4)} &= (1,\ 0,\ 2)\ . \end{aligned} \tag{4.3}$$

We find that $a^{(4)} = b^{(4)}$, so that

$$t = 4;\ t + 1 = 5 \;. \tag{4.4}$$

From (4.3) we calculate easily, in virtue of (2.4)

$$\begin{aligned} A_0^{(5)} &= 4;\ A_0^{(6)} = 5;\ A_0^{(7)} = 24;\ A_0^{(8)} = 53\,. \\ A_1^{(5)} &= 9;\ A_1^{(6)} = 11;\ A_1^{(7)} = 53;\ A_1^{(8)} = 117\,. \\ A_2^{(5)} &= 16;\ A_2^{(6)} = 20;\ A_2^{(7)} = 95;\ A_2^{(8)} = 209\ . \\ A_3^{(5)} &= 23;\ A_3^{(6)} = 28;\ A_3^{(7)} = 136;\ A_3^{(8)} = 300\,. \end{aligned} \tag{4.5}$$

Since here $(t + 1)(n + 1) = 5.3 = 15$, the determinant (3.54) is of the following form

$$\begin{vmatrix} 4 & 5 & 24 & 53 \\ 9 & 11 & 53 & 117 \\ 16 & 20 & 95 & 209 \\ 23 & 28 & 136 & 300 \end{vmatrix} = -1 \tag{4.6}$$

from which we obtain, developing D_5 in elements of the last column

$$53 \cdot 3 + 117 \cdot 3 + 209 \cdot (-1) + 300 \cdot (-1) = 1.$$

A solution vector of (4.1) is, therefore, given by

$$X = (3,\ 3,\ -1,\ -1)\ . \tag{4.7}$$

Since X is a standard solution vector, there is not need to transform (4.1) into an S. E. 4.

Let the S'. E. 4 have the form

$$37x + 89y + 131z + 401u = 1. \tag{4.8}$$

Proceeding as before, we obtain for the D_{t+1} of (3.54)

$$\begin{vmatrix} 1 & 2 & 7 & 37 \\ 2 & 5 & 17 & 89 \\ 3 & 7 & 25 & 131 \\ 10 & 22 & 76 & 401 \end{vmatrix} = 1\ , \tag{4.9}$$

which gives the solution vector for (4.8)

$$X = (-6,\ -2,\ 0,\ +1) \tag{4.10}$$

Since this vector has a zero among its components, we have to transform the S'. E. 4 of (4.8) into an S. E. 4. Here we choose

(4.11) $P = 2 \cdot 3 \cdot 5 \cdot 7;\ x = 30x';\ y = 42y';\ z = 70z';\ u = 105u'$.

Now the S'. E. 4 takes the form of an S. E. 4, viz.

(4.12) $$1110x' + 3738y' + 9170z' + 42105u' = 1 .$$

Carrying out the algorithm (2.8) of the given vector

(4.13) $$a^{(0)} = (3738/1110,\ 9170/1110, 42105/1110)$$

we obtain the vectors $b^{(v)}$

(4.14) $$\begin{aligned} &b^{(0)} = (3,\ 8,\ 37);\ b^{(1)} = (0,\ 2,\ 2);\ b^{(2)} = (0,\ 1,\ 1)\ ;\\ &b^{(3)} = (0,\ 0,\ 1);\ b^{(4)} = (29,\ 17,\ 54);\ b^{(5)} = (1,\ 1,\ 2)\ ;\\ &b^{(6)} = (1,\ 0,\ 2)\ . \end{aligned}$$

Here

$$t = 6,\quad t + 1 = 7,\quad (t + 1)(n - 1) = 21,\ D_7 = -1\ ;$$

after calculating the $A_i^{(v)}$, the determinant D_7 from (3.54) becomes

(4.15) $$\begin{vmatrix} 3 & 272 & 552 & 1110 \\ 10 & 916 & 1859 & 3738 \\ 25 & 2247 & 4560 & 9170 \\ 114 & 10318 & 20930 & 42105 \end{vmatrix} = -1\ ,$$

which gives the standard solution vector of (4.12)

(4.16) $$X' = (198,\ -23,\ -10,\ -1)\ ,$$

and, in view of (4.11) the standard solution vector of (4.8)

(4.17) $$X = (5940,\ -966,\ -700,\ -105)\ .$$

Let the S'E. 5 be

$$73x + 199y + 471z + 800u + 2001v = 1 \, . \tag{4.18}$$

Proceeding as before, we obtain for the determinant (3.54)

$$\begin{vmatrix} 4 & 21 & 21 & 22 & 73 \\ 11 & 57 & 57 & 60 & 199 \\ 26 & 136 & 135 & 142 & 471 \\ 44 & 230 & 230 & 241 & 800 \\ 110 & 576 & 576 & 603 & 2001 \end{vmatrix} = 1 \tag{4.19}$$

which gives the vector solution

$$X = (0, \; -2, \; 0, \; 3, \; -1) \; . \tag{4.20}$$

Since this vector has zero components, we have to transform the S'. E. 5 (4.18) into an S. E. 5. Here we choose

$$P = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11; \; x = 210x'; \; y = 330y'; \; z = 462z'; \; y = 770u'; \; v = 1155v' \; . \tag{4.21}$$

The S. E. 5 takes the form

$$15330x' + 65670y' + 217602z' + 616000u' + 2311155v' = 1 \, . \tag{4.22}$$

Carrying out the algorithm of the given vector

$$a^{(0)} = \left(\frac{65670}{15330}, \frac{217602}{15330}, \frac{616000}{15330}, \frac{2311155}{15330} \right) . \tag{4.23}$$

we obtain the vectors $b^{(v)}$

$$\begin{aligned} b^{(0)} &= (4, 14, 40, 150); \; b^{(1)} = (0, 0, 2, 3); \; b^{(2)} = (0, 0, 0, 1) \; ; \\ b^{(3)} &= (1, 0, 0, 1); \quad b^{(4)} = (14, 8, 1, 18); \; b^{(5)} = (1, 0, 0, 1) \; ; \\ b^{(6)} &= (1, 0, 2, 6); \quad b^{(7)} = (1, 1, 0, 2); \; b^{(8)} = (0, 2, 0, 9) \, . \end{aligned} \tag{4.24}$$

Here

$$t = 8, \quad t+1 = 9, \quad (t+1)(n-1) = 36;$$

after calculating the $A_i^{(v)}$, the determinant D_9 from (3.54) takes the form

$$(4.25) \qquad \begin{vmatrix} 95 & 99 & 790 & 1681 & 15330 \\ 407 & 424 & 3384 & 7201 & 65670 \\ 1349 & 1405 & 11213 & 23861 & 217602 \\ 3818 & 3978 & 31744 & 67547 & 616000 \\ 14323 & 14925 & 119100 & 253428 & 2311155 \end{vmatrix} = 1 ,$$

which gives the standard solution vectors of (4.22) and (4.18)

$$(4.26) \qquad X' = (1053, \ 26, \ -2, \ 13, \ -11) ,$$

$$(4.27) \qquad X = (221130, \ 8580, \ -924, \ 10010, \ -12705) .$$

5. THE CONJUGATE STANDARD EQUATIONS

DEFINITION. The Diophantine equations

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = c_1^{(v)} , \qquad (v = 1,\ldots,t-1) ;$$

$$c_j \text{ from } (1.2) , \qquad (j = 1,\ldots,n) ;$$

$$c_1^{(v)} \text{ from } (3.11); \ t \text{ from Theorem 3.1.}$$

will be called Conjugate Standard Equations.

In this chapter we shall find a solution vector for a conjugate standard equation and prove, to this end,

Theorem 5.1. A solution vector of the conjugate standard equation (5.1) is given by the vector whose j^{th} component is

$$(5.2) \qquad x_j = (-1)^{(v+1)(n-1)}B_{j,n}^{(v+1)} , \qquad (v = 1,\cdots, t-1)$$

where the $B_{j,n}^{(v+1)}$ are the cofactors of the elements in the n^{th} row of the determinant

$$(5.3)\qquad \begin{vmatrix} A_0^{(v+1)} & A_0^{(v+2)} & \cdots & A_0^{(v+n-1)} & c_1 \\ A_1^{(v+1)} & A_1^{(v+2)} & \cdots & A_1^{(v+n-1)} & c_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ A_{n-1}^{(v+1)} & A_{n-1}^{(v+1)} & \cdots & A_{n-1}^{(v+n-1)} & c_n \end{vmatrix}$$

If $(x_1^{(0)}, x_2^{(0)}, \cdots, x_n^{(0)})$ is a solution vector of the standard equation

$$c_1x_1 + c_2x_2 + \cdots + c_n x_n = 1 ,$$

then (5.2) is different from

$$(\cdots, x_j, \cdots) = (\cdots, x_j^{(0)}c_1^{(0)}, \cdots)(j = 1, 2, \cdots, n) .$$

Proof. As can be easily verified from the proof of Theorem 3.1, the relation holds

$$(5.4)\qquad a_{n-1}^{(v)} = c_1^{(v-1)} \Big/ c_1^{(v)} , \qquad (v = 1, 2, \cdots); \quad c_1^{(0)} = c_1 .$$

We shall first prove the formula

$$(5.5)\qquad A_0^{(v)} + \sum_{j=1}^{n-1} a_j^{(v)} A_0^{(v+j)} = a_{n-1}^{(1)} a_{n-1}^{(2)} \cdots a_{n-1}^{(v)}, \ (v=1,2,\cdots) .$$

We obtain, for $v = 1$, in view of (2.4),

$$A_0^{(1)} + \sum_{j=1}^{n-1} a_j^{(1)} A_0^{(1+j)} = a_{n-1}^{(1)} A_0^{(n)} = a_{n-1}^{(1)} ,$$

so that formula (5.5) is correct for $v = 1$. Let it be correct for $v = k$, viz.

$$A_0^{(k)} + \sum_{j=1}^{n-1} a_j^{(k)} A_0^{(k+j)} = a_{n-1}^{(1)} a_{n-1}^{(2)} \cdots a_{n-1}^{(k)}, \quad (k = 1, 2, \cdots). \tag{5.6}$$

From (2.3) we obtain

$$\begin{aligned} a_j^{(k)} &= \left(a_{j-1}^{(k+1)} \Big/ a_{n-1}^{(k+1)}\right) + b_j^{(k)}, \quad (j = 2, \cdots, n-1; k = 1, 2, \cdots) \\ a_1^{(k)} &= \left(1 \Big/ a_{n-1}^{(k+1)}\right) + b_1^{(k)} \end{aligned} \tag{5.7}$$

Rearranging the left side of the (5.6) by substituting there for $a_j^{(k)}$ the values from (5.7), we obtain

$$A_0^{(k)} + a_1^{(k)} A_0^{(k+1)} + \sum_{j=2}^{n-1} \left(\frac{a_{j-1}^{(k+1)} A_0^{(k+j)}}{a_{n-1}^{(k+1)}} + b_j^{(k)} A_0^{(k+j)} \right)$$

$$= a_{n-1}^{(1)} a_{n-1}^{(2)} \cdots a_{n-1}^{(k)} ;$$

The left side of this equation has the form

$$A_0^{(k)} + \frac{A_0^{(k+1)}}{a_{n-1}^{(k+1)}} + b_1 A_0^{(k+1)} + \sum_{j=2}^{n-1} \left(\frac{a_{j-1}^{(k+1)} A_0^{(k+j)}}{a_{n-1}^{(k+1)}} \right) + \sum_{j=2}^{n-1} b_j^{(k)} A_0^{(k+j)}$$

$$= \frac{A_0^{(k+1)} + \Sigma_{j=2}^{n-1} a_{j-1}^{(k+1)} \; A_0^{(k+j)}}{a_{n-1}^{(k+1)}} + \left(A_0^{(k)} + \sum_{j=1}^{n-1} b_j^{(k)} A_0^{(k+j)} \right)$$

$$= \left(A_0^{(k+1)} + \sum_{j=2}^{n-1} a_{j-1}^{(k+1)} A_0^{(k+1)} \right) \Big/ a_{n-1}^{(k+1)} + A_0^{(k+n)} = \left(A_0^{(k+1)} + \sum_{j=1}^{n-1} a_j^{(k+1)} A_0^{(k+1+j)} \right.$$

$$\left. + a_{n-1}^{k+1} A_0^{(k+n)} \right) \Big/ a_{n-1}^{(k+1)} = \left(A_0^{(k+1)} + \sum_{j=1}^{n-1} a_j^{(k+1)} A_0^{(k+1+j)} \right) \Big/ a_{n-1}^{(k+1)} .$$

We thus obtain

$$\left(A_0^{(k+1)} + \sum_{j=1}^{n-1} a_j^{(k+1)} A_0^{(k+1+j)}\right) \Big/ a_{n-1}^{(k+1)} = a_{n-1}^{(1)} a_{n-1}^{(2)} \cdots a_{n-1}^{(k)} ,$$

or

$$A_0^{(k+1)} + \sum_{j=1}^{n-1} a_j^{(k+1)} A_0^{(k+1+j)} = a_{n-1}^{(1)} a_{n-1}^{(2)} \cdots a_{n-1}^{(k+1)} . \tag{5.8}$$

But (5.8) is (5.5) for $v = k + 1$, which proves (5.5). From (5.4), (5.5), we now obtain

$$A_0^{(v)} + \sum_{j=1}^{n-1} a_j^{(v)} A_0^{(v+j)} = \frac{c_1}{c_1^{(1)}} \cdot \frac{c_1^{(1)}}{c_2^{(1)}} \cdots \frac{c_1^{(v-1)}}{c_1^{(v)}} ,$$

$$A_0^{(v)} + \sum_{j=1}^{n-1} a_j^{(v)} A_0^{(v+j)} = c_1 \Big/ c_1^{(v)} , \quad (v = 1, 2, \cdots) . \tag{5.9}$$

The reader should note that (5.9) holds for $v = 0$, too. We shall now return to formula (2.6. a), viz.

$$\begin{vmatrix} 1 & A_0^{(v+1)} & \cdots & A_0^{(v+n-1)} \\ a_1^{(0)} & A_1^{(v+1)} & \cdots & A_1^{(v+n-1)} \\ a_2^{(0)} & A_2^{(v+1)} & \cdots & A_2^{(v+n-1)} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n-1}^{(0)} & A_{n-1}^{(v+1)} & \cdots & A_{n-1}^{(v+n-1)} \end{vmatrix} = \frac{(-1)^{v(n-1)}}{A_0^{(v)} + \sum_{j=1}^{n-1} a_j^{(v)} A_0^{(v+j)}}$$

Substituting here the values of $a_j^{(0)}$ from (3.2) and for

$$A_0^{(v)} + \sum_{j=1}^{n-1} a_j^{(v)} A_0^{(v+j)}$$

from (5.9), we obtain

$$\begin{vmatrix} 1 & A_0^{(v+1)} & \cdots & A_0^{(v+n-1)} \\ c_2/c_1 & A_1^{(v+1)} & \cdots & A_1^{(v+n-1)} \\ c_3/c_1 & A_2^{(v+1)} & \cdots & A_2^{(v+n-1)} \\ \cdots & \cdots & \cdots & \cdots \\ c_n/c_1 & A_{n-1}^{(v+1)} & \cdots & A_{n-1}^{(v+n-1)} \end{vmatrix} = \frac{(-1)^{v(n-1)}}{c_1/c_1^{(v)}} .$$

or, multiplying both sides by c_1 and interchanging the first and the last row of the determinant,

$$(5.10) \qquad \begin{vmatrix} A_0^{(v+1)} & A_0^{(v+2)} & \cdots & A_0^{(v+n-1)} & c_1 \\ A_1^{(v+1)} & A_1^{(v+2)} & \cdots & A_1^{(v+n-1)} & c_2 \\ A_2^{(v+1)} & A_2^{(v+2)} & \cdots & A_2^{(v+n-1)} & c_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{n-1}^{(v+1)} & A_{n-1}^{(v+2)} & \cdots & A_{n-1}^{(v+n-1)} & c_n \end{vmatrix} = (-1)^{(v+1)(n-1)} c_1^{(v)} .$$

From (5.10) we obtain

$$c_1 B_{1,n}^{(v+1)} + c_2 B_{2,n}^{(v+1)} + \cdots + c_n B_{n,n}^{(v+1)} = (-1)^{(v+1)(n-1)} c_1^{(v)} ,$$

or, multiplying both sides by $(-1)^{(v+1)(n-1)}$

$$\sum_{j=1}^{n} c_j(-1)^{(v+1)(n-1)} B_{j,n}^{(v+1)} = c_1^{(v)} . \tag{5.11}$$

(5.11) proves the first statement of Theorem (5.1). To prove the second statement, we have to show that $c_1^{(v)}$ cannot be a divisor of all the

$$x_j = (-1)^{(v+1)(n-1)} B_{j,n}^{(v+1)} , \qquad (j = 1, \cdots, n)$$

To prove this, we recall formula (2.5), viz.

$$D_{v+1} = (-1)^{(v+1)(n-1)} , \tag{5.12}$$

so that

$$A_0^{(v+n)} B_{1,n}^{(v+1)} + A_1^{(v+n)} B_{2,n} + \cdots + A_{n-1}^{(v+n)} B_{n,n}^{(v+1)} = (-1)^{(v+1)(n-1)},$$

or

$$A_0^{(v+n)} x_1 + A_1^{(v+n)} x_2 + \cdots + A_{n-1}^{(v+n)} x_n = 1 . \tag{5.13}$$

From (5.13) we obtain

$$(x_1, x_2, \cdots, x_n) = 1 , \tag{5.14}$$

and since $c_1^{(v)} > 1$ for $v < t$, the second statement of Theorem 5.1 is proved. It should be stressed that the case

$$c_1^{(v_1)} = c_1^{(v_2)} = \cdots = c_1^{(v_k)}$$

is possible $(1 < k < t)$. In this case we shall consider the conjugate equations $c_1x_1 + c_2x_2 + \cdots + c_nx_n = c_1^{(v_j)}$, $(j = 1, \cdots, k)$ as different ones, since each of them will provide a different solution of (5.1) for the same $c_1^{(v)}$.

We shall solve some conjugate standard equations of (4.12), viz.

$$1110x' + 3738y' + 9170z' + 4210u' = 1 .$$

We calculate easily

$$(5.15) \qquad c_1^{(1)} = 408;\ c_1^{(2)} = 290;\ c_1^{(3)} = 219;\ c_1^{(4)} = 4;\ c_1^{(5)} = 2;\ t = 6.$$

Calculating the $A_i^{(v)}$ on basis of (4.14) we obtain a solution of

$$1110x' + 3738y' + 9170z' + 42105u' = 219 , \quad (v = 3)$$

$$X' = (-31, \ -2, \ 0, \ 1) \quad .$$

Similarly we obtain a solution of

$$1110x' + 3738y' + 9170z' + 42105u' = -4 \quad (v = 4)$$

$$X' = (-15, \ 2, \ 1, \ 0)$$

It should be well noted that the solution vectors of the conjugate standard equations are not necessarily standard solution vectors.

6. GENERALIZED FIBONACCI NUMBERS

The generalized Fibonacci numbers are defined by the initial values and the recursion formula as follows

$$(6.1) \qquad \begin{aligned} &F_1^{(n)} = F_2^{(n)} = \cdots = F_{n-1}^{(n)} = 0, \quad F_n^{(n)} = 1 ; \\ &F_{k+n}^{(n)} = \sum_{j=0}^{n-1} F_{k+j}^{(n)} ; \quad k + 1, n = 2, 3, \cdots . \end{aligned}$$

The numbers $F_i^{(n)}$ $(i = 1, 2, \cdots)$ will be called generalized Fibonacci numbers of degree n and order i. They are calculated by the generating function

(6.2) $$x^{n-1}\Big/(1 - x - x^2 - \cdots - x^n) = \sum_{i=1}^{\infty} F_i^{(n)} x^{i-1} .$$

Let denote

(6.3) $$f(x) = x^n + x^{n-1} + \cdots + x - 1 .$$

$f(x)$ from (6.3) is called the generating polynomial. This can be transformed into

(6.4) $$f(x) = (x^{n+1} - 2x + 1)/(x - 1), \quad x \neq 1 .$$

The equation

(6.5) $$(x - 1)f(x) = x^{n+1} - 2x + 1 = 0, \quad x \neq 1 ,$$

has 2 real roots and $(n - 2)/2$ pairs of conjugate complex roots for $n = 2m$ $(m = 1, 2, \cdots)$ and one real root and $(n - 1)/2$ pairs of conjugate complex roots for $n = 2m + 1$ $(m = 1, 2, \cdots)$. This is easily proved by analyzing the derivative of $f(x)$. The roots of $f(x)$ are, of course, irrationals. From (6.2) we obtain

(6.6) $$F_v^{(n)} = F_v^{(n)}(x_1, x_2, \cdots, x_n) , \quad (v = 1, 2, \cdots)$$

where $F_v^{(n)}(x_1, x_2, \cdots, x_n)$ is a symmetric function of the n roots of $f(x)$. It will be a main result of the next chapter to find an explicit formula for the ratio

(6.7) $$\lim_{v \to \infty} F_{v+1}^{(n)} \Big/ F_v^{(n)} .$$

In the case of the original Fibonacci numbers, viz. $n = 2$, this is a well-known fact. As can be easily verified from (6.2), the $F_v^{(2)}$ have the form

(6.8) $$F_{m+1}^{(2)} = \left(\left(\frac{\sqrt{5} + 1}{2}\right)^m \Big/ \sqrt{5}\right) + (-1)^{m-1}\left(\left(\frac{\sqrt{5} - 1}{2}\right)^m \Big/ \sqrt{5}\right) , \quad (m = 0, 1, \cdots) .$$

From (6.8) we obtain easily

$$\lim_{m\to\infty} F^{(2)}_{m+1} \Big/ F^{(2)}_{m} = (\sqrt{5} + 1)/2 \ . \tag{6.9}$$

Of course, for generalized Fibonacci numbers, a limiting formula analogous to (6.9) can be given by infinite series, as will be solved in the next chapter. We shall use the notation

$$D^{(n)}_{v} = \begin{vmatrix} F^{(n)}_{v} & F^{(n)}_{v+1} & \cdots & F^{(n)}_{v+n-1} \\ F^{(n)}_{v+1} & F^{(n)}_{v+2} & \cdots & F^{(n)}_{v+n} \\ \cdot\cdot\cdot & \cdot\cdot\cdot & \cdot\cdot\cdot & \cdot\cdot\cdot \\ F^{(n)}_{v+n-1} & F^{(n)}_{v+n} & \cdots & F^{(n)}_{v+2n-2} \end{vmatrix} , \quad (v = 1, 2, \cdots) \ . \tag{6.10}$$

We shall prove the formula

$$D^{(n)}_{v} = (-1)^{(n(n-1)/2)+(v-1)(n-1)} \ . \tag{6.11}$$

Proof by induction. We obtain from (6.1)

$$D^{(n)}_{1} = \begin{vmatrix} F^{(n)}_{1} & F^{(n)}_{2} & & \cdots & & F^{(n)}_{n} \\ F^{(n)}_{2} & F^{(n)}_{3} & & \cdots & F^{(n)}_{n} & F^{(n)}_{n+1} \\ F^{(n)}_{3} & F^{(n)}_{4} & \cdots & F^{(n)}_{n} & F^{(n)}_{n+1} & F^{(n)}_{n+2} \\ \cdot\cdot\cdot & \cdot\cdot\cdot & \cdot\cdot\cdot & \cdot\cdot\cdot & \cdot\cdot\cdot & \cdot\cdot\cdot \\ F^{(n)}_{n} & F^{(n)}_{n+1} & & \cdots & & F^{(n)}_{2n-1} \end{vmatrix} =$$

$$= \begin{vmatrix} 0 & 0 & & \cdots & & 1 \\ 0 & 0 & & 1 & F^{(n)}_{n+1} & \\ 0 & 0 & \cdots & 1 & F^{(n)}_{n+1} & F^{(n)}_{n+2} \\ \cdot\cdot\cdot & \cdot\cdot\cdot & \cdot\cdot\cdot & \cdot\cdot\cdot & \cdot\cdot\cdot & \cdot\cdot\cdot \\ 1 & F^{(n)}_{n+1} & & \cdots & & F^{(n)}_{2n-1} \end{vmatrix} ,$$

(6.12) $$D_1^{(n)} = (-1)^{n(n-1)/2} \quad , \quad (n = 2, 3, \cdots) \; .$$

We further obtain from (6.1)

$$D_v^{(n)} = \begin{vmatrix} F_v^{(n)} & F_{v+1}^{(n)} & \cdots & F_{v+n-2}^{(n)} & F_{v+n-1}^{(n)} \\ F_{v+1}^{(n)} & F_{v+2}^{(n)} & \cdots & F_{v+n-1}^{(n)} & F_{v+n}^{(n)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ F_{v+n-1}^{(n)} & F_{v+n}^{(n)} & \cdots & F_{v+2n-3}^{(n)} & F_{v+2n-2}^{(n)} \end{vmatrix} =$$

$$\begin{vmatrix} F_v^{(n)} & F_{v+1}^{(n)} & \cdots & F_{v+n-2}^{(n)} & (F_{v-1}^{(n)} + \sum_{j=1}^{n-1} F_{v-1+j}^{(n)}) \\ F_{v+1}^{(n)} & F_{v+2}^{(n)} & \cdots & F_{v+n-1}^{(n)} & (F_{v}^{(n)} + \sum_{j=1}^{n-1} F_{v+j}^{(n)}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ F_{v+n-1}^{(n)} & F_{v+n}^{(n)} & \cdots & F_{v+2n-3}^{(n)} & (F_{v+n-2}^{(n)} + \sum_{j=1}^{n-1} F_{v+n-2+j}^{(n)} \end{vmatrix} =$$

$$\begin{vmatrix} F_v^{(n)} & F_{v+1}^{(n)} & \cdots & F_{v+n-2}^{(n)} & F_{v-1}^{(n)} \\ F_{v+1}^{(n)} & F_{v+2}^{(n)} & \cdots & F_{v+n-1}^{(n)} & F_{v}^{(n)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ F_{v+n-1}^{(n)} & F_{v+n}^{(n)} & \cdots & F_{v+2n-3}^{(n)} & F_{v+n-2}^{(n)} \end{vmatrix} =$$

$$(-1)^{n-1} \begin{vmatrix} F_{v-1}^{(n)} & F_v^{(n)} & F_{v+1}^{(n)} & \cdots & F_{v+n-2}^{(n)} \\ F_v^{(n)} & F_{v+1}^{(n)} & F_{v+2}^{(n)} & \cdots & F_{v+n-1}^{(n)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ F_{v+n-2}^{(n)} & F_{v+n-1}^{(n)} & F_{v+n}^{(n)} & \cdots & F_{v+2n-3} \end{vmatrix} .$$

We have thus proved the formula

(6.13) $$D_v^{(n)} = (-1)^{n-1} D_{v-1}^{(n)} \; .$$

From (6.13) we obtain

$$D_v^{(n)} = (-1)^{n-1} D_{v-1}^{(n)} = (-1)^{n-1}(-1)^{n-1} D_{v-2} = \cdots = (-1)^{(v-1)(n-1)} D_1^{(n)}$$

which, together with (6.12), proves (6.11). We have simultaneously proved

Theorem 6.1. A vector solution of the S'. E. n

$$F_{v+n-1}^{(n)} x_1 + F_{v+n}^{(n)} x_2 + \cdots + F_{v+2n-2}^{(n)} x_n = 1 \tag{6.14}$$

is given by the formula

$$x_i = (-1)^{(n(n-1)/2)+(v-1)(n-1)} B_{i,n}, \quad (i = 1, \cdots, n), \tag{6.15}$$

where the $B_{i,n}$ are the cofactors of the elements in the n^{th} row of the determinant (6.10).

We shall now turn to the periodicity of the algorithm for ratios of cubic Fibonacci numbers and prove

Theorem 6.2. The Jacobi-Perron algorithm of the two irrationals

$$a_1^{(0)} = \lim_{v \to \infty} \left(F_{v+3}^{(3)} \middle/ F_{v+2}^{(3)}\right); \; a_2^{(0)} = \lim_{v \to \infty} \left(F_{v+4}^{(3)} \middle/ F_{v+2}^{(3)}\right) \tag{6.16}$$

is periodic; the preperiod has the length $S = 2$ and the form

$$\begin{matrix} 1 & 3 & , \\ 0 & 1 & . \end{matrix} \tag{6.17}$$

The period has the length $T = 6$ and the form

$$\begin{matrix} 0 & 2 & , \\ 0 & 2 & , \\ 0 & 2 & , \\ 0 & 2 & , \\ 0 & 1 & , \\ 0 & 4 & , \\ 0 & 1 & . \end{matrix} \tag{6.18}$$

<u>Proof.</u> We shall first prove the following inequalities

$$F_{v+3}^{(3)} < F_{v+4}^{(3)} < 2F_{v+3}^{(3)} \;, \quad (v = 3, 4, \cdots) \; ; \tag{6.19}$$

$$3F_{v+2}^{(3)} < F_{v+4}^{(3)} < 4F_{v+2}^{(3)} \;, \quad v \text{ as above } . \tag{6.20}$$

From

$$F_{v+4}^{(3)} = F_{v+3}^{(3)} + F_{v+2}^{(3)} + F_{v+1}^{(3)} \; ; \; F_{v+1}^{(3)}, \, F_{v+2}^{(3)} > 0 \quad \text{for} \quad v \geq 2 \; ,$$

we obtain

$$F_{v+4}^{(3)} > F_{v+3}^{(3)} \; .$$

We further obtain

$$F_{v+4}^{(3)} = 2F_{v+3}^{(3)} - (F_{v+3}^{(3)} - F_{v+2}^{(3)} - F_{v+1}^{(3)}) \; ,$$

but

$$F_{v+3}^{(3)} - F_{v+2}^{(3)} - F_{v+1}^{(3)} = F_{v}^{(3)} > 0, \quad \text{for} \;\; v = 3, 4, \cdots$$

therefore

$$F_{v+4}^{(3)} < 2F_{v+3}^{(3)} \; ,$$

which proves (6.19). We further obtain

$$\begin{aligned} F_{v+4}^{(3)} &= F_{v+3}^{(3)} + F_{v+2}^{(3)} + F_{v+1}^{(3)} \\ &= (F_{v+2}^{(3)} + F_{v+1}^{(3)} + F_{v}^{(3)}) + F_{v+2}^{(3)} + F_{v+1}^{(3)} \\ &= 2F_{v+2}^{(3)} + 2F_{v+1}^{(3)} + F_{v}^{(3)} \\ &= 2F_{v+2}^{(3)} + (F_{v+1}^{(3)} + F_{v}^{(3)} + F_{v-1}^{(3)}) + F_{v+1}^{(3)} - F_{v-1}^{(3)} \\ &= 3F_{v+2}^{(3)} + F_{v+1}^{(3)} - F_{v-1}^{(3)} \; ; \end{aligned}$$

but

$$F^{(3)}_{v+1} - F^{(3)}_{v-1} = F^{(3)}_{v} + F^{(3)}_{v-2} > 0 \text{ for } v \geq 3 ,$$

therefore

$$F^{(3)}_{v+4} > 3F^{(3)}_{v+2} .$$

Since

$$F^{(3)}_{v+2} = F^{(3)}_{v+1} + F^{(3)}_{v} + F^{(3)}_{v-1} = 2F^{(3)}_{v} + 2F^{(3)}_{v-1} + F^{(3)}_{v-2} > F^{(3)}_{v} + F^{(3)}_{v-2}$$

for $v \geq 3$, we obtain

$$F^{(3)}_{v+1} - F^{(3)}_{v-1} = F^{(3)}_{v} + F^{(3)}_{v-2} < F^{(3)}_{v+2} ,$$

and, therefore, from the previous result

$$F^{(3)}_{v+4} < 4F^{(3)}_{v+2} ,$$

which proves (20).

We shall now carry out the algorithm of Jacobi-Perron for the numbers

(6.21) $$a_1^{(0)} = F^{(3)}_{v+3} \Big/ F^{(3)}_{v+2} \; ; \; a_2^{(0)} = F^{(3)}_{v+4} \Big/ F^{(3)}_{v+2} , \quad v \geq 12 .$$

Though the proof is carried out for the rationals

$$F^{(3)}_{v+3} \Big/ F^{(3)}_{v+2} \quad \text{and} \quad F^{(3)}_{v+4} \Big/ F^{(3)}_{v+2} ,$$

and not for their limiting values, the reader will understand, after having read Chapter 7, that this is permissible.

We obtain from (6.19), substituting $v - 1$ for v, and in virtue of $v \geq 12$,

$$F^{(3)}_{v+2} < F^{(3)}_{v+3} < 2F^{(3)}_{v+2} \;;\; 1 < F^{(3)}_{v+3} \Big/ F^{(3)}_{v+2} < 2 \;,$$

so that

$$b_1^{(0)} = [a_1^{(0)}] = 1 \;. \tag{6.22}$$

From (6.20), we obtain

$$3 < F^{(3)}_{v+4} \Big/ F^{(3)}_{v+2} < 4 \;,$$

so that

$$b_2^{(0)} = [a_2^{(0)}] = 3 \;. \tag{6.23}$$

From (6.21), (6.22), (6.23), we obtain

$$\begin{aligned} a_2^{(1)} &= 1 \Big/ (a_1^{(0)} - b_1^{(0)}) = 1 \Big/ \left((F^{(3)}_{v+3} \Big/ F^{(3)}_{v+2}) - 1 \right) \\ &= F^{(3)}_{v+2} \Big/ (F^{(3)}_{v+3} - F^{(3)}_{v+2}) = F^{(3)}_{v+2} \Big/ (F^{(3)}_{v+1} + F^{(3)}_{v}) \;; \end{aligned}$$

$$\begin{aligned} a_1^{(1)} &= (a_2^{(0)} - b_2^{(0)}) \Big/ (a_1^{(0)} - b_1^{(0)}) \\ &= \left(\frac{F^{(3)}_{v+4}}{F^{(3)}_{v+2}} - 3 \right) \frac{F^{(3)}_{v+2}}{F^{(3)}_{v+1} + F^{(3)}_{v}} = (F^{(3)}_{v+4} - 3F^{(3)}_{v+2} \Big/ (F^{(3)}_{v+1} + F^{(3)}_{v}) \;; \end{aligned}$$

but, as has been proved before,

$$F^{(3)}_{v+4} - 3F^{(3)}_{v+2} = F^{(3)}_{v} + F^{(3)}_{v-2} \;;$$

we thus obtain

$$a_1^{(1)} = \frac{F^{(3)}_{v} + F^{(3)}_{v-2}}{F^{(3)}_{v+1} + F^{(3)}_{v}} \;;\; a_2^{(1)} = \frac{F^{(3)}_{v+2}}{F^{(3)}_{v+1} + F^{(3)}_{v}} \;. \tag{6.24}$$

Since

$$0 < F_v^{(3)} + F_{v-2}^{(3)} < F_{v+1}^{(3)} + F_v^{(3)} ,$$

we obtain

$$0 < (F_v^{(3)} + F_{v-2}^{(3)})/(F_{v+1}^{(3)} + F_v^{(3)}) < 1 ;$$

since further

$$F_{v+2}^{(3)} / (F_{v+1}^{(3)} + F_v^{(3)}) = (F_{v+1}^{(3)} + F_v^{(3)} + F_{v-1}^{(3)}) / (F_{v+1}^{(3)} + F_v^{(3)}) =$$

$$1 + (F_{v-1}^{(3)} / (F_{v+1}^{(3)} + F_v^{(3)})), \text{ and since } F_{v-1}^{(3)} < F_{v+1}^{(3)} + F_v^{(3)} ,$$

we obtain

$$b_1^{(1)} = 0; \quad b_2^{(1)} = 1 . \tag{6.25}$$

From (6.24), (6.25), we obtain

$$1/(a_1^{(1)} - b_1^{(1)}) = (F_{v+1}^{(3)} + F_v^{(3)}) / (F_v^{(3)} + F_{v-2}^{(3)}) ;$$

$$a_2^{(1)} - b_2^{(1)} = (F_{v+2}^{(3)} - F_{v+1}^{(3)} - F_v^{(3)}) / (F_{v+1}^{(3)} + F_v^{(3)}) =$$

$$F_{v-1}^{(3)} / (F_{v+1}^{(3)} + F_v^{(3)}) ;$$

we thus obtain, in virtue of (2.3)

$$a_1^{(2)} = \frac{F_{v-1}^{(3)}}{F_v^{(3)} + F_{v-2}^{(3)}} ; \quad a_2^{(2)} = \frac{F_{v+1}^{(3)} + F_v^{(3)}}{F_v^{(3)} + F_{v-2}^{(3)}} . \tag{6.26}$$

From (6.26) we obtain, since

$$0 < F_{v-1}^{(3)} < F_v^{(3)} + F_{v-2}^{(3)} , \quad 0 < F_{v-1}^{(3)} / (F_v^{(3)} + F_{v-2}^{(3)}) < 1 ,$$

and further, since

$$(F_{v+1}^{(3)} + F_v^{(3)}) / (F_v^{(3)} + F_{v-2}^{(3)}) = (2F_v^{(3)} + F_{v-1}^{(3)} + F_{v-2}^{(3)}) / (F_v^{(3)} + F_{v-2}^{(3)}) =$$
$$= (2F_v^{(3)} + 2F_{v-2}^{(3)} + F_{v-3}^{(3)} + F_{v-4}^{(3)}) / (F_v^{(3)} + F_{v-2}^{(3)}) =$$
$$= 2 + ((F_{v-3}^{(3)} + F_{v-4}^{(3)})/(F_v^{(3)} + F_{v-2}^{(3)})) < 3 ,$$

so that

$$b_1^{(2)} = 0; \quad b_2^{(2)} = 2 . \tag{6.27}$$

From (6.26), (6.27), we obtain, on basis of the previous results

$$1 / (a_1^{(2)} - b_1^{(2)}) = (F_v^{(3)} + F_{v-2}^{(3)}) / F_{v-1}^{(3)} ;$$

$$a_2^{(2)} - b_2^{(2)} = ((F_{v+1}^{(3)} + F_v^{(3)}) / (F_v^{(3)} + F_{v-2}^{(3)})) - 2 = (F_{v-3}^{(3)} + F_{v-4}^{(3)}) / (F_v^{(3)} + F_{v-2}^{(3)});$$

we thus obtain, in virtue of (2.3),

$$a_1^{(3)} = \frac{F_{v-3}^{(3)} + F_{v-4}^{(3)}}{F_{v-1}^{(3)}} ; \quad a_2^{(3)} = \frac{F_v^{(3)} + F_{v-2}^{(3)}}{F_{v-1}^{(3)}} . \tag{6.28}$$

Since

$$F_{v-3}^{(3)} + F_{v-4}^{(3)} < F_{v-3}^{(3)} + F_{v-4}^{(3)} + F_{v-2}^{(3)} = F_{v-1}^{(3)} ,$$

we obtain

$$b_1^{(3)} = [a_1^{(3)}] = 0 .$$

We further obtain

$$F_v^{(3)} + F_{v-2}^{(3)} = F_{v-1}^{(3)} + 2F_{v-2}^{(3)} + F_{v-3}^{(3)}$$
$$= F_{v-1}^{(3)} + (F_{v-2}^{(3)} + F_{v-3}^{(3)} + F_{v-4}^{(3)}) + F_{v-2}^{(3)} - F_{v-4}^{(3)}$$
$$= 2F_{v-1}^{(3)} + F_{v-2}^{(3)} - F_{v-4}^{(3)} ;$$

Therefore

(6.29)
$$2F_{v-1}^{(3)} < F_v^{(3)} + F_{v-2}^{(3)} < 3F_{v-1}^{(3)} \; ; \; 2 < \frac{F_v^{(3)} + F_{v-2}^{(3)}}{F_{v-1}^{(3)}} < 3 \; ;$$
$$b_1^{(3)} = 0 \; ; \qquad b_2^{(3)} = 2 \; .$$

From (6.28), (6.29), we obtain

$$1 \Big/ (a_1^{(3)} - b_1^{(3)}) = F_{v-1}^{(3)} / (F_{v-3}^{(3)} + F_{v-4}^{(3)}) \; ;$$
$$a_2^{(3)} - b_2^{(3)} = ((F_v^{(3)} + F_{v-2}^{(3)}) \Big/ F_{v-1}^{(3)}) - 2$$
$$= (F_{v-2}^{(3)} - F_{v-4}^{(3)}) \Big/ F_{v-1}^{(3)} = (F_{v-3}^{(3)} + F_{v-5}^{(3)} \Big/ F_{v-1}^{(3)} ,$$

so that, in virtue of (2.3),

(6.30)
$$a_1^{(4)} = \frac{F_{v-3}^{(3)} + F_{v-5}^{(3)}}{F_{v-3}^{(3)} + F_{v-4}^{(3)}} \; ; \quad a_2^{(4)} = \frac{F_{v-1}^{(3)}}{F_{v-3}^{(3)} + F_{v-4}^{(3)}} \; .$$

From (6.30) we obtain

$$b_1^{(4)} = [a_1^{(4)}] = 0 \; ,$$

and further

$$F_{v-1}^{(3)} = F_{v-2}^{(3)} + F_{v-3}^{(3)} + F_{v-4}^{(3)} = 2(F_{v-3}^{(3)} + F_{v-4}^{(3)}) + F_{v-5}^{(3)} \; ,$$

so that

$$F_{v-1}^{(3)} \Big/ (F_{v-3}^{(3)} + F_{v-4}^{(3)}) = 2 + (F_{v-5}^{(3)} \Big/ (F_{v-3}^{(3)} + F_{v-4}^{(3)})) \; ,$$

or

$$2 < (F^{(3)}_{v-1} / (F^{(3)}_{v-3} + F^{(3)}_{v-4})) < 3 ,$$

which finally yields

$$b_1^{(4)} = 0; \; b_2^{(4)} = 2 . \tag{6.31}$$

From (6.30), (6.31), we obtain

$$1 / (a_1^{(4)} - b_1^{(4)}) = (F^{(3)}_{v-3} + F^{(3)}_{v-4}) / (F^{(3)}_{v-3} + F^{(3)}_{v-5}) ;$$

$$a_2^{(4)} - b_2^{(4)} = F^{(3)}_{v-5} / (F^{(3)}_{v-3} + F^{(3)}_{v-4}) ,$$

so that, in virtue of (2.3),

$$a_1^{(5)} = \frac{F^{(3)}_{v-5}}{F^{(3)}_{v-3} + F^{(3)}_{v-5}} , \qquad a_2^{(5)} = \frac{F^{(3)}_{v-3} + F^{(3)}_{v-4}}{F^{(3)}_{v-3} + F^{(3)}_{v-5}} . \tag{6.32}$$

From (6.32) we obtain

$$[a_1^{(5)}] = b_1^{(5)} = 0 ,$$

and further,

$$(F^{(3)}_{v-3} + F^{(3)}_{v-4}) / (F^{(3)}_{v-3} + F^{(3)}_{v-5})$$

$$= (F^{(3)}_{v-3} + F^{(3)}_{v-5} + F^{(3)}_{v-6} + F^{(3)}_{v-7})) / (F^{(3)}_{v-3} + F^{(3)}_{v-5}) = 1 + \frac{F^{(3)}_{v-6} + F^{(3)}_{v-7}}{F^{(3)}_{v-3} + F^{(3)}_{v-5}} ,$$

so that

$$1 < ((F^{(3)}_{v-3} + F^{(3)}_{v-4}) / (F^{(3)}_{v-3} + F^{(3)}_{v-5})) < 2 ,$$

which yields

(6.33) $$b_1^{(5)} = 0; \quad b_2^{(5)} = 1 .$$

From (6.23), (6.33), we obtain easily

(6.34) $$a_1^{(6)} = \frac{F_{v-6}^{(3)} + F_{v-7}^{(3)}}{F_{v-5}^{(3)}}; \quad a_2^{(6)} = \frac{F_{v-3}^{(3)} + F_{v-5}^{(3)}}{F_{v-5}^{(3)}} .$$

From (6.34) we obtain

$$b_1^{(6)} = [a_1^{(6)}] = 0 ,$$

and further

$$\begin{aligned} F_{v-3}^{(3)} + F_{v-5}^{(3)} &= F_{v-4}^{(3)} + 2F_{v-5}^{(3)} + F_{v-6}^{(3)} = 3F_{v-5}^{(3)} + 2F_{v-6}^{(3)} + F_{v-7}^{(3)} \\ &= 3F_{v-5}^{(3)} + (F_{v-6}^{(3)} + F_{v-7}^{(3)} + F_{v-8}^{(3)}) + F_{v-6}^{(3)} - F_{v-8}^{(3)} \\ &= 4F_{v-5}^{(3)} + F_{v-7}^{(3)} + F_{v-9}^{(3)} < 4F_{v-5}^{(3)} < 4F_{v-6}^{(3)} + 5F_{v-5}^{(3)} ; \end{aligned}$$

therefore,

$$4 < \left((F_{v-3}^{(3)} + F_{v-5}^{(3)}) \Big/ F_{v-5}^{(3)} \right) < 5 ,$$

so that

(6.35) $$b_1^{(6)} = 0; \quad b_2^{(6)} = 4 .$$

From (6.34), (6.35), we obtain

$$1 / (a_1^{(6)} - b_1^{(6)}) = (F_{v-5}^{(3)} \Big/ (F_{v-6}^{(3)} + F_{v-7}^{(3)}) ,$$

$$a_2^{(6)} - b_2^{(6)} = (F_{v-7}^{(3)} + F_{v-9}^{(3)}) \Big/ F_{v-5}^{(3)} ,$$

so that, in virtue of (2.3)

(6.36) $$a_1^{(7)} = \frac{F_{v-7}^{(3)} + F_{v-9}^{(3)}}{F_{v-6}^{(3)} + F_{v-7}^{(3)}} ; \quad a_2^{(7)} = \frac{F_{v-5}^{(3)}}{F_{v-6}^{(3)} + F_{v-7}^{(3)}} .$$

From (6.36) we obtain

$$b_1^{(7)} = [a_1^{(7)}] = 0 ,$$

and further

$$F_{v-5}^{(3)} \Big/ (F_{v-6}^{(3)} + F_{v-7}^{(3)}) = (F_{v-6}^{(3)} + F_{v-7}^{(3)} + F_{v-8}^{(3)}) \Big/ (F_{v-6}^{(3)} + F_{v-7}^{(3)})$$
$$= 1 + (F_{v-8}^{(3)} \Big/ (F_{v-6}^{(3)} + F_{v-7}^{(3)})) ,$$

so that

$$(6.37) \qquad b_1^{(7)} = 0 ; \quad b_2^{(7)} = 1 .$$

From (6.36, (6.37), we obtain

$$1 \Big/ (a_1^{(7)} - b_1^{(7)}) = (F_{v-6}^{(3)} + F_{v-7}^{(3)}) \Big/ (F_{v-7}^{(3)} + F_{v-9}^{(3)}) ,$$
$$a_2^{(7)} - b_2^{(7)} = F_{v-8}^{(3)} \Big/ (F_{v-6}^{(3)} + F_{v-7}^{(3)}) ,$$

so that, in virtue of (2.3),

$$(6.38) \qquad a_1^{(8)} = \frac{F_{v-8}^{(3)}}{F_{v-7}^{(3)} + F_{v-9}^{(3)}} ; \quad a_2^{(8)} = \frac{F_{v-6}^{(3)} + F_{v-7}^{(3)}}{F_{v-7}^{(3)} + F_{v-9}^{(3)}} .$$

Substituting in (6.38) for v the value

$$(6.39) \qquad v = u + 7 ,$$

we obtain

$$(6.40) \qquad a_1^{(8)} = \frac{F_{u-1}^{(3)}}{F_u^{(3)} + F_{u-2}^{(3)}} ; \quad a_2^{(8)} = \frac{F_{u+1}^{(3)} + F_u^{(3)}}{F_u^{(3)} + F_{u-2}^{(3)}} .$$

Comparing (6.26) with (6.40), we see that

$$(6.41) \qquad a_1^{(8)} = a_1^{(2)} \; ; \quad a_2^{(8)} = a_2^{(2)} \quad \text{for} \quad u = v \to +\infty \; ,$$

which proves the first statement of Theorem 6.2. The forms of the preperiod (6.17) and the period (6.18) is verified by the formulas (6.22) and (6.23, 25,···, 35, 37).

Applying Theorem (5.1) to the Jacobi-Perron algorithm of the numbers

$$F_{v+3}^{(3)} \Big/ F_{v+2}^{(3)} \; , \quad F_{v+4}^{(3)} \Big/ F_{v+2}^{(3)}$$

(this Theorem holds for any algorithm (2.3), as long as the formation law of the $b_i^{(v)}$ generates integers) and singling out the denominators

$$\begin{aligned} c_1^{(2)} &= F_v^{(3)} + F_{v-2}^{(3)} \; , \\ c_1^{(3)} &= F_{v-1}^{(3)} \; , \\ c_1^{(4)} &= F_{v-3}^{(3)} + F_{v-4}^{(3)} \; , \end{aligned}$$

we obtain, on ground of (6.41) and the vector equations $a^{(9)} = a^{(3)}$, $a^{(10)} = a^{(4)}$,

$$(6.42) \qquad \begin{aligned} c_1^{(2+6k)} &= F_{v-7k}^{(3)} + F_{v-2-7k}^{(3)} \; , \\ c_1^{(3+6k)} &= F_{v-1-7k} \; , \\ c_1^{(4+6k)} &= F_{v-3-7k}^{(3)} + F_{v-4-7k}^{(3)} \; . \end{aligned}$$

From (6.42), we obtain, in virtue of (5.3), where $n = 3$,

$$(6.43) \qquad \begin{vmatrix} A_0^{(3+6k)} & A_0^{(4+6k)} & F_{v+2}^{(3)} \\ A_1^{(3+6k)} & A_1^{(4+6k)} & F_{v+3}^{(3)} \\ A_2^{(3+6k)} & A_2^{(4+6k)} & F_{v+4}^{(3)} \end{vmatrix} = F_{v-7k}^{(3)} + F_{v-2-7k}^{(3)}, \quad v \geq 7k + 3.$$

Substituting in (6.43) $v = u + 7k$, we obtain that a solution vector of the S'.E.3

$$(6.44) \qquad xF_{u+2+7k}^{(3)} + yF_{u+3+7k}^{(3)} + zF_{u+4+7k}^{(3)} = F_u^{(3)} + F_{u-2}^{(3)} \; ,$$

$$k = 0, 1, \cdots \; ; \quad u = 3, 4, \cdots$$

is given by

$$(6.45)\qquad \begin{aligned} x &= A_1^{(3+6k)} A_2^{(4+6k)} - A_1^{(4+6k)} A_2^{(3+6k)} , \\ y &= A_2^{(3+6k)} A_0^{(4+6k)} - A_2^{(4+6k)} A_0^{(3+6k)} , \\ z &= A_0^{(3+6k)} A_1^{(4+6k)} - A_0^{(4+6k)} A_1^{(3+6k)} . \end{aligned}$$

Substituting in (6.44) $u = 5$, we obtain that (6.45) is a solution vector of

$$(6.46)\qquad xF^{(3)}_{7(k+1)} + yF^{(3)}_{7(k+1)+1} + zF^{(3)}_{7(k+2)+2} = 3 .$$

We further obtain from (6.42), in virtue of (5.3),

$$(6.47)\qquad \begin{vmatrix} A_0^{(4+6k)} & A_0^{(5+6k)} & F^{(3)}_{v+2} \\ A_1^{(4+6k)} & A_1^{(5+6k)} & F^{(3)}_{v+3} \\ A_2^{(4+6k)} & A_2^{(5+6k)} & F^{(3)}_{v+4} \end{vmatrix} = F^{(3)}_{v-1-7k}$$

Substituting in (6.47) $v = u + 7k$, we obtain that a solution vector of the S'.E.3

$$(6.48)\qquad xF^{(3)}_{u+2+7k} + yF^{(3)}_{u+3+7k} + zF^{(3)}_{u+4+7k} = F^{(3)}_{u-1} , \quad k = 0, 1, \cdots ;\ u = 4, 5, \cdots ,$$

is given by

$$(6.49)\qquad \begin{aligned} x &= A_1^{(4+6k)} A_2^{(5+6k)} - A_1^{(5+6k)} A_2^{(4+6k)} ; \\ y &= A_2^{(4+6k)} A_0^{(5+6k)} - A_2^{(5+6k)} A_0^{(4+6k)} ; \\ z &= A_0^{(4+6k)} A_1^{(5+6k)} - A_0^{(5+6k)} A_1^{(4+6k)} . \end{aligned}$$

We obtain from (6.48), for $u = 6$, that the equation

$$(6.50)\qquad xF^{(3)}_{7(k+1)+1} + yF^{(3)}_{7(k+1)+2} + zF^{(3)}_{7(k+1)+3} = 2$$

has the vector solution (6.49).

We further obtain from (6.42), in virtue of (5.3)

$$(6.51)\qquad \begin{vmatrix} A_0^{(5+6k)} & A_0^{(6+6k)} & F_{v+2}^{(3)} \\ A_1^{(5+6k)} & A_1^{(6+6k)} & F_{v+3}^{(3)} \\ A_2^{(5+6k)} & A_2^{(6+6k)} & F_{v+4}^{(3)} \end{vmatrix} = F_{v-3-7k}^{(3)} + F_{v-4-7k}^{(3)} .$$

Substituting in (6.51) $v = u + 7k$, we obtain that a solution vector of

$$(6.52)\qquad xF_{u+2+7k}^{(3)} + yF_{u+3+7k}^{(3)} + zF_{u+4+7k}^{(3)} = F_{u-3}^{(3)} + F_{u-4}^{(3)} ;$$
$$k = 0, 1, \cdots ; \quad u = 6, 7, \cdots .$$

is given by

$$(6.53)\qquad x = A_1^{(5+6k)}A_2^{(6+6k)} - A_1^{(6+6k)}A_2^{(5+6k)}; \; y = A_2^{(5+6k)}A_0^{(6+6k)} - A_2^{(6+6k)}A_0^{(5+6k)}$$
$$z = A_0^{(5+6k)}A_1^{(6+6k)} - A_0^{(6+6k)}A_1^{(5+6k)} .$$

We obtain from (6.52), for $u = 9$, that a solution vector of

$$(6.54)\qquad xF_{7(k+1)+4}^{(3)} + yF_{7(k+1)+5}^{(3)} + zF_{7(k+1)+6}^{(3)} = 6$$

is given by (6.53).

We shall give a few numeric examples for this theory. If we put $k = 1$ in (6.50), we obtain

$$xF_{15}^{(3)} + yF_{16}^{(3)} + zF_{17}^{(3)} = 2 .$$

From (6.49), we calculate easily

$$x = -20; \quad y = -2; \quad z = 7$$

so that

$$(6.55)\qquad 7F_{17}^{(3)} - 2F_{16}^{(3)} - 20F_{15}^{(3)} = 2 .$$

We calculate easily

$$F_{15}^{(3)} = 927;\ F_{16}^{(3)} = 1705;\ F_{17}^{(3)} = 3136,$$

which verifies (6.55).

If we put $k = 1$ in (6.54), we obtain

$$xF_{18}^{(3)} + yF_{19}^{(3)} + zF_{20}^{(3)} = 6 .$$

From (6.53), we calculate easily

$$x = -38;\quad y = -29;\quad z = 27 ,$$

so that

$$27F_{20}^{(3)} - 29F_{19}^{(3)} - 38F_{18}^{(3)} = 6 . \tag{6.56}$$

We calculate easily

$$F_{18}^{(3)} = 5768;\quad F_{19}^{(3)} = 10609;\quad F_{20}^{(3)} = 19513$$

which verifies (6.56).

7. THE GENERATING POLYNOMIAL OF GENERALIZED FIBONACCI NUMBERS

The main purpose of this chapter will be the statement of an explicit formula for the limiting value of the ratio

$$F_{v-1}^{(n)} \Big/ F_{v}^{(n)}$$

of two successive generalized Fibonacci numbers of degree $n \geq 2$. To this end, we shall investigate the generating polynomial $f(x)$ from (6.3) recalling a few results of the author stated in a previous paper [1. p)]. We obtain from (6.3)

$$f(0) = -1; \quad f(1) = n - 1 > 0 ;$$

$$f'(x) = \sum_{k=0}^{n-1} (n - k)x^{n-1-k} > 0 \quad \text{for } x > 0.$$

Therefore $f(x)$ has one and only one real root w in the open interval $(0, 1)$, so that

$$w^n + w^{n-1} + \cdots + w - 1 = 0; \quad 0 < w < 1 . \tag{7.1}$$

We shall now carry out the modified Jacobi-Perron algorithm of the numbers

$$a_s^{(0)} = \sum_{i=0}^{s} w^{s-i} , \quad (s = 1,\cdots,n - 1) , \tag{7.2}$$

which are the components of the given vector $a^{(0)}$. These have, therefore, the form of (7.2), viz.

$$a_1^{(0)} = w + 1; \; a_2^{(0)} = w^2 + w + 1; \; \cdots; \; a_{n-1}^{(0)} = w^{n-1} + w^{n-2} + \cdots + 1.$$

Then the numbers $a_s^{(v)}$ are functions of w, viz.

$$a_s^{(v)} = a_s^{(v)}(w) , \quad (s = 1,\cdots,n - 1; v = 0,1,\cdots) . \tag{7.3}$$

For the formation law of the rationals $b_s^{(v)}$ we use the formation law

$$b_s^{(v)} = a_s^{(v)}(0) , \quad (s = 1,\cdots,n - 1; v = 0,1,\cdots) . \tag{7.4}$$

The author has proved in [1, p)] that under these assumptions the modified Jacobi-Perron algorithm of the given vector (6.2) is purely periodic; the length of the period is $T = 1$, and it has the form

$$b_s^{(v)} = 1, \quad (s = 1,\cdots,n - 1; v = 0,1,\cdots) . \tag{7.5}$$

As has been proved by the author in 1. p) , the formula holds

$$w = \lim_{v \to \infty} \left(A_0^{(v-1)} \middle/ A_0^{(v)} \right) , \tag{7.6}$$

where the $A_0^{(v)}$ have the meaning of (2.4). From (2.4) and (7.5), we obtain

$$\begin{aligned} A_0^{(0)} &= 1 , \\ A_0^{(1)} &= 0 = F_1^{(n)} , \\ A_0^{(2)} &= 0 = F_2^{(n)} , \\ &\cdots\cdots\cdots \\ A_0^{(n-1)} &= 0 = F_{n-1}^{(n)} . \end{aligned}$$

Since

$$A_0^{(n)} = A_0^{(0)} + \sum_{j=1}^{n-1} b_j^{(0)} A_0^{(j)} = 1 + \sum_{j=1}^{n-1} A_0^{(j)} = 1 ,$$

we have

$$A_0^{(n)} = F_n^{(n)} = 1 .$$

We have thus obtained

$$A_0^{(i)} = F_i^{(n)} , \quad (v = 1, 2, \cdots) . \tag{7.7}$$

We shall now prove that (7.7) holds for any $i \geq 1$, viz.

$$A_0^{(v)} = F_v^{(n)} , \quad (v = 1, 2, \cdots) . \tag{7.8}$$

Proof by induction. In virtue of (7.7) formula (7.8) is correct for $v = 1, 2, \cdots, n$. Let (7.8) be correct for

$$v = k, k+1, \cdots, k+(n-1) , \quad k \geq 1 \tag{7.9}$$

We shall now prove that (7.8) is correct for $v = k + n$. We obtain from (2.4) and (7.5), (7.9)

$$A_0^{(k+n)} = A_0^{(k)} + \sum_{j=1}^{n-1} b_j^{(k)} A_0^{(k+j)}$$

$$= A_0^{(k)} + \sum_{j=1}^{n-1} A_0^{(k+j)}$$

$$= F_k^{(n)} + \sum_{j=1}^{n-1} F_{k+j}^{(n)} = F_{k+n}^{(n)} ,$$

which proves formula (7.8).

Combining (7.6) and (7.8), we obtain the formula

$$w = \lim_{v\to\infty} (F_{v-1}^{(n)} / F_v^{(n)}) . \tag{7.10}$$

Theoretically (7.10) is a very significant formula and answers the questions posed in (6.7). But practically it is of no great value, since neither w nor $F_v^{(n)}$ can be calculated easily because of lack of an explicit formula for either of them. This problem will be solved in the forthcoming passages.

The polynomial $x^{n+1} - 2x + 1$, $x \neq 1$, has the same roots as the generating polynomial $f(x) = x^n + x^{n-1} + \cdots + x - 1$. Particularly, it has one, and only one, real root in the open interval $(0,1)$, viz. w from (7.1). In a previous paper [1. p)] the author has proved the following

<u>Theorem.</u> Let be

$$F(w) = w^{n+1} - 2w + 1 = 0 , \quad 0 < w < 1 . \tag{7.11}$$

If we carry out the modified algorithm of Jacobi-Perron for the given vector $a^{(0)}$ with the components

(7.12) $$a_s^{(0)} = w^s\ , \quad (s = 1,\cdots,n-1);\ a_n^{(0)} = w^n - 2,$$

then the algorithm becomes purely periodic; the length of the period is $T = n + 1$, and it has the form

(7.13) $$n+1\left\{\begin{matrix} 0 & \overbrace{0 & \cdots & 0}^{n} & -2 \\ 0 & 0 & \cdots & 0 & 2 \\ 0 & 0 & \cdots & 0 & 2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & 2 \end{matrix}\right.$$

If, for $v > v_0$,

(7.14) $$\frac{\left|A_0^{(v)}\right| + \sum_{j=1}^{n-1} \left|a_j^{(0)}\right| \left|A_0^{(v+j)}\right|}{\left|a_n^{(0)}\right| \left|A_0^{(v+n)}\right|} \le m < 1\ ,$$

then

(7.15) $$w = \lim\ (A_0^{(v-1)} \Big/ A_0^{(v)})\ .$$

We thus have only to prove that (7.14) holds for the modified algorithm of Jacobi-Perron of (7.12). We obtain from (2.14) and (7.13)

$$A_0^{(0)} = 1;\ A_0^{(v)} = 0,\ (v = 1,\cdots,n);\ A_0^{(n+1)} = 1;$$

(7.16) $$A_0^{(n+2)} = A_0^{(1)} + \sum_{j=1}^{n} b_j^{(1)} A_0^{(1+j)} = b_n^{(1)} A_0^{(n+1)} = 2\ ;$$

$$A_0^{(n+3)} = A_0^{(2)} + \sum_{j=1}^{n} b_j^{(2)} A_0^{(2+j)} = b_n^{(2)} A_0^{(n+2)} = 2^2\ .$$

We shall now prove

$$A_0^{(n+1+v)} = 2^v, \quad (v = 0, 1, \cdots, n) . \tag{7.17}$$

Proof by induction. (7.17) is correct for $v = 0, 1, 2,$ in virtue of (7.16). Let it be correct for $v = k$, viz.

$$A_0^{(n+1+k)} = 2^k, \quad (k = 0, 1, \cdots, n-1) . \tag{7.18}$$

From (7.18) we obtain

$$\begin{aligned} A_0^{(n+1+k+1)} &= A_0^{(k+1)} + \sum_{j=1}^{n} b_j^{(k+1)} A_0^{(k+1+j)} = b_n^{(k+1)} A_0^{(n+1+k)} \\ &= 2 \cdot 2^k = 2^{k+1} , \end{aligned}$$

which proves (7.17). We further obtain from (7.16), (7.17)

$$\begin{aligned} A_0^{(n+1+n+1)} &= A_0^{(n+1)} + \sum_{j=1}^{n} b_j^{(n+1)} A_0^{(n+1+j)} \\ &= 2 + b_n^{(n+1)} A_0^{(n+1+n)} = 2 + b_n^{(0)} A_0^{(n+1+j)} \\ &= 2 + (-2) \cdot 2^n = 2 - 2^{n+1}; \left|A_0^{(n+1+n+1)}\right| \geq \frac{2n+1}{n+1} \cdot 2^n, \quad n \geq 3, \end{aligned}$$

$$\left|A_0^{(n+1+n+1)}\right| \geqq \frac{2n+1}{n+1} \cdot \left|A_0^{(n+1+n)}\right| . \tag{7.19}$$

We now deduce from (7.17), (7.19),

$$\left|A_0^{(n+1+v)}\right| \geq \frac{2n+1}{n+1}\left|A_0^{(n+v)}\right| \text{ for } v = 0, 1, \cdots, n+1 \tag{7.20}$$

and shall prove generally

$$\left|A_0^{(n+1+v)}\right| > \frac{2n+1}{n+1}\left|A_0^{(n+v)}\right| , \quad (v = 0, 1, \cdots) . \tag{7.21}$$

Proof by induction. Let be

$$\left|A_0^{(n+1+v)}\right| \geq \frac{2n+1}{n+1}\left|A_0^{(n+v)}\right|, \quad \text{for } v = k, k+1, \cdots, k+n-1. \tag{7.22}$$

(7.22) is correct for $k = 0, 1, 2,$ in virtue of (7.20). We now obtain, in virtue of (2.4), (7.13),

$$\begin{aligned} A_0^{(n+1+k+n)} &= A_0^{(k+n)} + \sum_{j=1}^{n} b_j^{(n+k)} A_0^{(k+n+j)} \\ &= A_0^{(k+n)} + b_n^{(k+n)} A_0^{(k+n+n)} \\ &= A_0^{(k+n)} \pm 2A_0^{(k+n+n)}, \end{aligned}$$

$$\left|A_0^{(n+1+k+n)}\right| \geq 2\left|A_0^{(k+n+n)}\right| - \left|A_0^{(k+n)}\right|. \tag{7.23}$$

But from (7.22) we obtain

$$\begin{aligned} \left|A_0^{(n+k)}\right| &\leq \frac{n+1}{2n+1}\left|A_0^{(n+k+1)}\right| \leq \left(\frac{n+1}{2n+1}\right)^2 \left|A_0^{(n+k+2)}\right| \\ &\cdots \leq \left(\frac{n+1}{2n+1}\right)^n \left|A_0^{(k+n+n)}\right|, \end{aligned}$$

$$\left|A_0^{(k+n)}\right| \leq \left(\frac{n+1}{2n+1}\right)^n \left|A_0^{(k+n+n)}\right|. \tag{7.24}$$

From (7.23), (7.24) we obtain

$$\left|A_0^{(n+1+k+n)}\right| \geq \left(2 - \left(\frac{n+1}{2n+1}\right)^n\right)\left|A_0^{(k+n+n)}\right|. \tag{7.25}$$

We shall now prove

$$2 - \left(\frac{n+1}{2n+1}\right)^n > \frac{2n+1}{n+1}, \quad \text{for } n = 3, 4, \cdots \tag{7.26}$$

We have to prove

$$2 - \left(\frac{n+1}{2n+1}\right)^n > 2 - \frac{1}{n+1} , \quad \text{or} \quad n + 1 < \left(\frac{2n+1}{n+1}\right)^n \quad \text{or}$$

$$n + 1 < \left(1 + \frac{n}{n+1}\right)^n , \quad n = 3, 4, \cdots .$$

But, for $n \geq 3$,

$$1 + \binom{n}{1} \cdot \frac{n}{n+1} + \binom{n}{2}\left(\frac{n}{n+1}\right)^2 < \left(1 + \frac{n}{n+1}\right)^n .$$

We shall prove

$$n + 1 \leq 1 + \binom{n}{1} \cdot \frac{n}{n+1} + \binom{n}{2}\left(\frac{n}{n+1}\right)^2 ,$$

or

$$n \leq \frac{n^2}{n+1} + \frac{n^3(n-1)}{2(n+1)^2} ,$$

or

$$1 \leq \frac{n}{n+1} + \frac{n^2(n-1)}{2(n+1)^2} ,$$

or

$$\frac{1}{n+1} \leq \frac{n^2(n-1)}{2(n+1)^2} ; \quad 2(n+1) \leq n^2(n-1) .$$

But, for $n \geq 3$,

$$n^2(n-1) \geq 2n^2 \geq 6n = 2n + 4n \geq 2n + 12 > 2n + 2$$
$$= 2(n+1) .$$

Thus (7.26) is proved.

From (7.25), (7.26), we obtain

$$\left|A_0^{(n+1+k+n)}\right| > \frac{2n+1}{n+1}\left|A_0^{(k+n+n)}\right| ,$$

which proves (7.21).

From (7.21) we obtain

$$(7.27)\qquad \left|A_0^{(k+v)}\right| > \left(\frac{2n+1}{n+1}\right)^k \left|A_0^{(v)}\right| , \quad (k+v \geq n+1) .$$

We shall now prove formula (7.14). We obtain, since

$$\left|a_j^{(0)}\right| = w^j < 1 , \qquad (j = 1, \cdots, n-1) ;$$
$$\left|a_n^{(0)}\right| = 2 - w^n , \qquad n \geq 3 ,$$

$$\frac{\left|A_0^{(v)}\right| + \sum_{j=1}^{n-1} \left|a_j^{(0)}\right| \left|A_0^{(v+j)}\right|}{\left|a_n^{(0)}\right| \left|A_0^{(v+n)}\right|} < \frac{\sum_{j=0}^{n-1} \left|A_0^{(v+j)}\right|}{(2 - w^n)\left|A_0^{(v+n)}\right|} .$$

But from (7.22) we obtain

$$\left|A_0^{(v+j)}\right| < \left(\frac{n+1}{2n+1}\right)^{n-j} \left|A_0^{(v+n)}\right| ,$$

therefore

$$\frac{\left|A_0^{(v)}\right| + \sum_{j=1}^{n-1} \left|a_j^{(0)}\right| \left|A_0^{(v+j)}\right|}{\left|a_n^{(0)}\right| \left|A_0^{(v+n)}\right|} < \frac{\sum_{j=0}^{n-1} \left(\frac{n+1}{2n+1}\right)^{n-j}}{2 - w^n}$$

$$= \frac{\frac{n+1}{2n+1}\left(1 - \left(\frac{n+1}{2n+1}\right)^n\right)}{\left(1 - \frac{n+1}{2n+1}\right)(2 - w^n)} = \frac{(n+1)\left(1 - \left(\frac{n+1}{2n+1}\right)^n\right)}{(2 - w^n)n} ,$$

so that

$$(7.28)\qquad \frac{\left|A_0^{(v)}\right| + \sum_{j=1}^{n-1} \left|a_j^{(0)}\right| \left|A_0^{(v+j)}\right|}{\left|a_n^{(0)}\right| \left|A_0^{(v+n)}\right|} < \frac{(n+1)}{(2 - w^n)n} \left(1 - \left(\frac{n+1}{2n+1}\right)^n\right)$$

We shall now prove

$$(7.29) \qquad (n+1)/n < 2 - w^n, \quad n \geq 3 .$$

We obtain from

$$F(x) = x^{n+1} - 2x + 1 ,$$

$$F(0) = 1, \quad F(1) = 0; \quad F'(x) = (n+1)x^n - 2 ;$$

therefore

$$F'(x) < 0 \quad \text{for} \quad 0 < x^n < 2/(n+1) ,$$

$$F'(x) > 0 \quad \text{for} \quad x^n > 2/(n+1) .$$

Since w is the only root in the open interval (0.1), we obtain

$$(7.30) \qquad w^n < \frac{2}{n+1} .$$

From (7.30) we obtain

$$2 - \frac{2}{n+1} < 2 - w^n .$$

It is easy to prove the following formula

$$\frac{n+1}{n} < 2 - \frac{2}{n+1} .$$

With (7.31) and the previous result (7.29) is proved. From (7.28), (7.29), we obtain

$$\frac{\left|A_0^{(v)}\right| + \sum_{j=1}^{n-1} \left|a_j^{(0)}\right| \left|A_0^{(v+j)}\right|}{\left|a_n^{(0)}\right| \left|A_0^{(v+n)}\right|} < 1 - \left(\frac{n+1}{2n+1}\right)^n . \tag{7.32}$$

But (7.32) verifies (7.14) with

$$m = 1 - \left(\frac{n+1}{2n+1}\right)^n < 1 . \tag{7.33}$$

We shall use a formula for the $A_0^{(v)}$ of an algorithm with the period (7.13) proved by the author in [1. p)], viz.

$$A_0^{((s+1)(n+1)+k)} = b^k \sum_{i=0}^{s} \binom{i(n+1)+s+k-i}{i(n+1)+k} z^i \tag{7.34}$$

$$b = 2; \quad z = -2^{n+1}; \quad (s = 0, 1, \cdots ; k = 0, 1, \cdots, n)$$

Writing in formula (7.15) $v = (s+1)(n+1)+1$, we obtain

$$w = \lim_{s\to\infty} \left(A_0^{((s+1)(n+1))} \Big/ A_0^{((s+1)(n+1)+1)}\right) ,$$

and, using (7.34),

$$w = \lim_{s\to\infty} \frac{\sum_{i=0}^{s} (-1)^i \binom{(n+1)i+s-i}{(n+1)i} 2^{(n+1)i}}{2\sum_{i=0}^{s} (-1)^i \binom{(n+1)i+s+1-i}{(n+1)i+1} 2^{(n+1)i}} . \tag{7.35}$$

Comparing (7.10) and (7.35), we obtain the wanted relation

$$\lim_{s\to\infty} \frac{F_{s-1}^{(n)}}{F_s^{(n)}} = \frac{1}{2} \lim_{s\to\infty} \frac{\sum_{i=0}^{s} (-1)^i \binom{(n+1)i+s-i}{(n+1)i} 2^{(n+1)i}}{\sum_{i=0}^{s} (-1)^i \binom{(n+1)i+s+1-i}{(n+1)i+1} 2^{(n+1)i}} . \tag{7.36}$$

REFERENCES

1. Leon Bernstein, a) "Periodical Continued Fractions of degree n by Jacobi's Algorithm," Journal f. d. Reine Angew. Math., Band 213, 1964, pp. 31-38.

b) "Representation of $(D^n - d)^{1/n}$ as a Periodic Continued Fraction by Jacobi's Algorithm," Math. Nachrichten, 1019, 1965, pp. 179-200.

c) "Periodicity of Jacobi's Algorithm for a Special Type of Cubic Irrationals," Journal f. d. Reine Angew. Math., Band 213, 1964, pp. 134-146.

d) "Period. Jacobische Algor. fuer eine unendliche Klasse Algebr. Irrationalzahlen vom Grade n etc.," Journal f. d. Reine Angew. Math., Band 215/216, 1944, pp. 76-83.

e) "Periodische Jacobi-Perronsche Algorithmen," Archiv der Math., Band XV, 1964, pp. 421-429.

f) "New Infinite Classes of Periodic Jacobi-Perron Algor.," Pacific Journal of Mathematics, Vol. 16, No. 3, 1966, pp. 439-469.

g) "A Period. Jacobi-Perron Algor.," The Canadian Journal of Math., 1965, Vol. 17, pp. 933-945.

Leon Bernstein and Helmut Hasse h) "Einheitenberechnung mittels d. Jacobi-Perronschen Algor.," Jour. f. d. Reine Angew. Math., Band 218, 1965, pp. 51-69.

Leon Bernstein i) "Rational Approx. of Algebr. Irrationals by Means of a Modified Jacobi-Perron Algor.," Duke Math. Journal, Vol. 32, No. 1, 1965, pp. 161-176.

REFERENCES (Cont'd.)

Leon Bernstein j) "Period. Kettenbrueche beliebiger Periodenlaenge," _Math. Zeitschrift_, Band 86, 1964, pp. 128-135.

k) "A Probability Function for Partitions," To appear soon.

l) "Rational Approximation of Algebr. Numbers," _Proceedings of the National Conference on Data Processing, Rehovoth_, 1964, IPA, pp. 91-105.

m) "Der B- Algorithms und Seine Anwendung," _Journal f. d. Reine Angew. Mathematik_, Band 227, 1967, pp. 150-177.

o) "The Generalized Pellian Equation," _Transaction of the American Math. Soc._, Vol. 127, No. 1, April 1967, pp. 76-89.

p) "The Modified Algorithm of Jacobi-Perron," _Memoirs_ of the American Math. Soc., Number 67, 1966, pp. 1-44.

q) "Ein Neuer Algorithmus fuer absteigende Basispotenzen im Kubischen Koerper," _Mathematische Nachrichten_, Band 33 (1967), Heft 5/6, pp. 257-272.

Leon Bernstein and Helmut Hasse r) "Maximal Number of Units in an Algebraic Number Field," to appear soon.

2. C. G. J. Jacobi, "Allgemeine Theorie der kettenbruchaehnlichen Algorithmen, in welchen jede Zahl aus drei vor = hergehenden gebildet wird," _Journal f. d. Reine Angew._ Math., 69 1968.

3. Oskar Perron, "Grundlagen fuer eine Theorie der Jacobischen Kettenbruchalgorithmen," _Mathematische Annalen_, 64 (1907).

* * * * *