RECURRING SEQUENCES—LESSON 1

BROTHER ALFRED BROUSSEAU
St. Mary's College, California

The Fibonacci Quarterly has been publishing an abundance of material over the past five years dealing in the main with the Fibonacci sequence and its relatives. Basic to the entire undertaking is the concept of RECURRING SEQUENCE. In view of this fact, a series of some eight lessons has been prepared covering this topic. In line with the word "lesson," examples of principles will be worked out in the articles and a number of "problems" will be included for the purpose of providing "exercise" in the material presented. Answers to these problems will be included on another page so that people may be able to check their work against them.

In this first lesson, the idea of sequence and recursion relation will be considered in a general way. A sequence is an ordered set of quantities. The sequence is finite if the set of quantities terminates; it is infinite if it does not. The prototype of all sequences is the sequence of positive integers: 1, 2, 3, 4, 5, $\cdots$. Other sequences, some quite familiar, are the following:

$$1,\ 3,\ 5,\ 7,\ 9,\ 11,\ 13,\ \cdots$$
$$2,\ 4,\ 6,\ 8,\ 10,\ 12,\ 14,\ 16,\ \cdots$$
$$1,\ 2,\ 4,\ 8,\ 16,\ 32,\ 64,\ \cdots$$
$$2,\ 6,\ 18,\ 54,\ 162,\ 486,\ \cdots$$
$$1,\ 2,\ 6,\ 24,\ 120,\ 720,\ 5040,\ 40320,\ \cdots$$
$$1,\ 3,\ 6,\ 10,\ 15,\ 21,\ 28,\ 36,\ 45,\ 55,\ \cdots$$
$$1,\ 4,\ 9,\ 16,\ 25,\ 36,\ 49,\ 64,\ \cdots\ .$$
$$1,\ 1/2,\ 1/3,\ 1/4,\ 1/5,\ 1/6,\ 1/7,\ 1/8,\ \cdots$$

For convenience of reference, the terms of sequences can be identified by the following notation: $a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ \cdots,\ a_n,\ \cdots$. One of the common ways of providing a compact representation of a sequence is to specify a formula for the n^{th} term. For the positive integers, $a_n = n$; for the odd integers 1, 3, 5, 7, $\cdots$, $a_n = 2n - 1$; for the even integers 2, 4, 6, 8, $\cdots$

$a_n = 2n$. The n^{th} terms of the remaining sequences given above are listed herewith.

$$1, 2, 4, 8, 16, 32, \cdots, a_n = 2^{n-1}$$
$$2, 6, 18, 54, 162, 486, \cdots, a_n = 2 \cdot 3^{n-1}$$
$$1, 2, 6, 24, 120, \cdots, a_n = n!$$
$$1, 3, 6, 10, 15, 21, 28, \cdots, a_n = n(n+1)/2$$
$$1, 4, 9, 16, 25, 36, \cdots, a_n = n^2$$
$$1, 1/2, 1/3, 1/4, \cdots, a_n = 1/n .$$

There is, however, a second way of specifying sequences and that is the recursion approach. The word recursion derives from recur and indicates that something is happening over and over. When in a sequence, there is an operation which enables us to find a subsequent term by using previous terms according to some well-defined method, we have what can be termed a recursion sequence. Again, the prototype is the sequence of positive integers which is completely specified by giving the first term $a_1 = 1$ and stating the recursion relation

$$a_{n+1} = a_n + 1 \quad .$$

This is the general pattern for a recursion sequence; one or more initial terms must be specified; then an operation (or operations) is set down which enables one to generate any other term of the sequence.

Going once more to some of our previous sequences, the recursion representations are as follows:

$$1, 3, 5, 7, \cdots, \; a_1 = 1; \quad a_{n+1} = a_n + 2 .$$
$$2, 4, 6, 8, \cdots, \; a_1 = 2 ; \quad a_{n+1} = a_n + 2.$$
$$1, 2, 4, 8, 16, \cdots, a_1 = 1; \quad a_{n+1} = 2a_n \; .$$
$$2, 6, 18, 54, 162, \cdots, a_1 = 2; \quad a_{n+1} = 3a_n \; .$$
$$1, 2, 6, 24, 120, \cdots, a_1 = 1; \quad a_{n+1} = (n+1)a_n \; .$$

Is it possible in all instances to give this dual interpretation to a sequence, that is, to specify the n^{th} term on the one hand and to provide a recursion definition of the sequence on the other? Is it not wise to say in an absolute manner what is possible or impossible in mathematics. But at least it can be stated that sequences which are readily representable by their n^{th} term may be difficult to represent by recursion and on the contrary, sequences which can be easily represented by recursion may not have an obvious n^{th} term. For example, what is the recursion relation for the sequence defined by:

$$a_n = \sqrt[n]{\frac{\log n}{\sqrt[3]{n}}} \qquad ?$$

Or on the other hand, if $a_1 = 2$, $a_2 = 3$, $a_3 = 5$, and

$$a_{n+1} = \frac{7a_n + 5a_{n-1}}{a_{n-2}}$$

what is the expression for the n^{th} term?

However, in most of the usual cases, it is possible to have both the n^{th} term and the recursion formulation of a sequence. Many of the common sequences, for example, have their n^{th} term expressed as a polynomial in n. In such a case, it is possible to find a corresponding recursion relation. In fact, for all polynomials of a given degree, there is just one recursion relation corresponding to them, apart from the initial values that are given. Let us examine this important case.

Our discussion will be based on what are known as finite differences. Given a polynomial in n, such as $f(n) = n^2 + 3n - 1$, we define

$$\Delta f(n) = f(n + 1) - f(n)$$

(Read "the first difference of $f(n)$" for $\Delta f(n)$.) Let us carry out this operation.

$$\Delta f(n) = (n + 1)^2 + 3(n + 1) - 1 - (n^2 + 3n - 1)$$

$$\Delta f(n) = 2n + 4 .$$

Note that the degree of $\Delta f(n)$ is one less than the degree of the original polynomial. If we take the difference of $\Delta f(n)$ we obtain the second difference of $f(n)$. Thus

$$\Delta^2 f(n) = 2(n+1) + 4 - (2n+4) = 2$$

Finally, the third difference of $f(n)$ is $\Delta^3 f(n) = 2 - 2 = 0$. The situation portrayed here is general. A polynomial of degree m has a first difference of degree $m - 1$, a second difference of degree $m - 2, \cdots$, an m^{th} difference which is constant and an $(m + 1)^{st}$ difference which is zero. Basically, this result depends on the lead term of highest degree. We need only consider then what happens to $f(n) = n^m$ when we take a first difference.

$$\Delta f(n) = (n+1)^m - n^m = n^m + mn^{m-1} \cdots -n^m$$

$\Delta f(n) = mn^{m-1} + \cdots$ terms of lower degree. Thus the degree drops by 1.

Suppose we designate the terms of our sequence as T_n. Then

$$\Delta T_n = T_{n+1} - T_n$$
$$\Delta^2 T_n = T_{n+2} - T_{n+1} - (T_{n+1} - T_n) = T_{n+2} - 2T_{n+1} + T_n$$
$$\Delta^3 T_n = T_{n+3} - 2T_{n+2} + T_{n+1} - (T_{n+2} - 2T_{n+1} + T_n)$$

or

$$\Delta^3 T_n = T_{n+3} - 3T_{n+2} + 3T_{n+1} - T_n .$$

Clearly the coefficients of the Pascal triangle with alternating signs are being generated and it is clear from the operation that this will continue.

We are now ready to transform a sequence with a term expressed as a polynomial in n into a recursion relation. Consider again:

$$T_n = n^2 + 3n - 1 .$$

Take the third difference of both sides. Then

$$\Delta^3 T_n = \Delta^3(n^2 + 3n - 1)$$

But the third difference of a polynomial of the second degree is zero. Hence

$$T_{n+3} - 3T_{n+2} + 3T_{n+1} - T_n = 0$$

or

$$T_{n+3} = 3T_{n+2} - 3T_{n+1} + T_n$$

is the required recursion relation for all sequences whose term can be expressed as a polynomial of the second degree in n.

An interesting particular case is the arithmetic progression whose n^{th} term is

$$T_n = a + (n - 1)d ,$$

where a is the first term and d the common difference. For example, if a is 5 and d is 4,

$$T_n = 5 + 4(n - 1) = 4n - 1 .$$

In any event, an arithmetic progression has a term which can be expressed as a polynomial of the first degree in n. Accordingly the recursion relation for all arithmetic progression is:

$$\Delta^2 T_n = 0$$

or

$$T_{n+2} = 2T_{n+1} - T_n .$$

The recursion relation for the geometric progression with ratio r is evidently

$$T_{n+1} = rT_n \ .$$

For example, 2, 18, 54, 162, $\cdots$ is specified by $a_1 = 2$, $T_{n+1} = 3T_n$.

This takes care of our listed sequences except the factorial and the reciprocal of n. For the factorial:

$$T_{n+1} = (n + 1)T_n \ .$$

However, we do not have a pure recursion relation to a subsequent from previous terms of the sequence. We need to eliminate n in the coefficient to bring this about. Now

$$n = T_n / T_{n-1}$$

and

$$n + 1 = T_{n+1} / T_n \ .$$

Thus

$$T_{n+1} / T_n - T_n / T_{n-1} = 1$$

so that

$$T_{n+1} = T_n (T_n + T_{n-1}) / T_{n-1} \ .$$

Again for $T_n = 1/n$, we have

$$n = 1/T_n \ , \quad n + 1 = 1/T_{n+1}, \quad 1/T_{n+1} - 1/T_n = 1$$

so that

$$T_{n+1} = T_n / (1 + T_n) \ .$$

PROBLEMS

1. Find the n^{th} term and the recursion relation for the sequence: 2, 6, 12, 20, 30, 42, 56, $\cdots$.
2. Find the n^{th} term and the recursion relation for the sequence: 1, 4, 7, 10, 13, 16, $\cdots$.
3. Determine the n^{th} term and the recursion relation for the sequence: 1, 8, 27, 64, 125, 216, 343, $\cdots$.
4. For $T_1 = 1$, $T_2 = 3$ and $T_{n+1} = T_n / T_{n-1}$, find a form of expression for the n^{th} term. (It may be more convenient to do this using a number of formulas.)
5. Find the recursion relation for the sequence with the term $T_n = \sqrt{n}$.
6. What is the recursion relation for a sequence whose term is a cubic polynomial in n?
7. If a is a positive constant, determine the recursion relation for the sequence with the term $T_n = a^n$.
8. Find a recursion relation corresponding to $T_{n+1} = T_n + 2n + 1$ which does not involve n except in the subscripts nor a constant except as a coefficient.
9. Find an expression(s) for the n^{th} term of the sequence the recursion relation $T_n T_{n+1} = 1$, where $T_1 = a$ (a not zero).
10. For the sequence with term $T_n = n/(n + 1)$, find a recursion relation with n occurring only in subscripts.

See page 260 for answers to problems.

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