FIBONACCI NUMBERS AND THE SLOW LEARNER

JAMES C. CURL
Tracy High School, Tracy, Calif.

Fibonacci numbers have been used with remarkable success with talented mathematics students from elementary school through graduate level university mathematics. They have been used as both a part of the basic curriculum and as enrichment material.

During the past year I became interested in the possibility of using Fibonacci numbers with a group of twenty-five freshman students in a "low level" basic mathematics class, the required ninth grade general mathematics course designed to fulfill the mathematics requirement for the freshman year. My interest in using Fibonacci numbers was a result of having met considerable frustration in trying to get the class to achieve a basic facility with the fundamental operations using real numbers. After a semester's work with review, explanations, and drill, the class still had difficulty with the same problems.

Before continuing, I should define "low level" to place the balance of my remarks in proper context. During the fall of this past academic year, the students in the class were given a battery of tests including the Differential Aptitude Test, the Gates Reading Survey, and the Lorge Thorndike Intelligence Quotient Test. I compiled a table of the scores on these three tests utilizing the verbal I. Q. score, the numerical ability percentile ranking, and the composite score on the Gates Reading Survey. The following information was compiled as a result of the research; two-thirds of the class had a numerical ability percentile ranking of fifteen percentile or lower. One-half of the class read at the sixth grade level or lower. One-half of the class had I. Q. scores of less than ninety. Only four students were reading at the ninth grade level or higher. Two students ranked above the fiftieth percentile on the numerical ability, and only four students had I. Q. scores of better than one hundred.

Irving Adler in his address at the California Mathematics Council, Northern Section, meeting at Davis during the spring of this year suggested that as teachers we are being a little unrealistic if by repeating the same material we believe we are able to do what competent teachers have failed to do during a student's first eight years. With much the same philosophy I decided to refine my goals for the course after the first semester. Basically, I hoped to:

(1) effect a change in attitude toward mathematics,

(2) have some of the students get excited over learning something,

(3) achieve basic skills in the four operations working with real numbers.

With the above goals in mind I introduced a five-week unit on Fibonacci numbers via a presentation on the board. Very little introduction was given: I simply announced that we would be working with something new, Fibonacci numbers. The class reaction was a collective, "WHAT ? ? ?". I then proceeded to write 1, 1, 2 on the board and asked the class to follow along as I wrote the next number down; they were to see if they could find out how I was getting the sequence of numbers.

After writing down several more terms of the sequence, the class caught on to the pattern. Within a very short time the entire class was volunteering the next number. We continued until we had the first twenty numbers written down. We then discussed how we could find the numbers of the sequence and ended the session with the simple explanation, "add the first two numbers and you get the third; add the second and third numbers and you get the fourth, $\cdots$" Although in the context of this article I will utilize a formal notation to express Fibonacci patterns, no attempt was made in class at this time to express the patterns with a general notation.

I next asked the students to write down on a paper the first fifty terms of the Fibonacci sequence. If they were not able to finish in class, they were welcome to do so at home. To my delight the majority of the class had worked on the first fifty and several had worked on getting the first one hundred Fibonacci numbers.

On the second day I handed out a ditto with the first one hundred Fibonacci numbers written down. After checking the values for their numbers, we discussed the notation $F_1, F_2, F_3, \cdots$ which I had used on the ditto. We called the notation the Fibonacci code for telling which Fibonacci number we were discussing. I encouraged each student to use the notation when he worked with a pattern.

During the first week, including the introduction, the class participated in what Brother Alfred terms group research. The class developed the pattern for

$$F_n^2 - F_{n+k}F_{n-k}$$

for $k = 1$ using the group research method. Instead of stating the problem in the preceding form, each person was asked to pick out some Fibonacci number and square it. Then they were to take the product of the two Fibonacci numbers on either side of the number that had been squared. Finally they were to find the difference between the square and the product. The results were tabulated on the board and the class was asked to try a different Fibonacci number. Again the results were written down. The majority of the students quickly saw that we were getting 1 for an answer; however, when I requested that the subtraction must be done in the same order each time a Fibonacci number had been selected, some of the students remarked that you couldn't subtract a larger number from a smaller one. We then had a delightful discussion about directed numbers and ended with the generalization that the answer was 1 if we chose a Fibonacci number with an odd code and -1 otherwise. I discussed $(-1)^{n+1}$ with the more capable students as a way of expressing the pattern.

The class then worked the next day on extending this pattern for different values of k. We started with group research again for $k = 2$ and found the difference of $|1|$. I then had the class work at their desks finding the patterns for other values of k, but not until I had encouraged them to make a conjecture about what they might find. It was a much surprised group of students when the next value of k did not give then $|1|$ for an answer. I was amused at their discovery that different values of k gave what appeared to them to be quite unrelated answers. Although the students became frustrated easily and I found it necessary to spend time helping each one, the problem allowed each student to continue at his own pace. After considerable work, one of the students suddenly saw that the result was a Fibonacci number squared. Some other students were finding this result and sharing their discovery with others around them. It was at this time that I felt I was achieving some of my goals.

The next problem presented was written down in the following way.

$$1 + 1 = 2; \quad 1 + 1 + 2 = 4; \quad 1 + 1 + 2 + 3 = 7; \cdots$$

I asked the students to write down the numbers in Fibonacci code, and the class was asked to find a pattern in the answers. It was necessary to give some direction by asking if the answer was close to some Fibonacci number. Finally we wrote down the result in the form

$$F_1 + F_2 + 1 = 2 + 1 = 3$$

$$F_1 + F_2 + F_3 + 1 = 4 + 1 = 5$$

I then asked for a verbal generalization from the students and it was decided that if one was added to each of the sums, we obtained a Fibonacci number. I asked them to give the Fibonacci code for the number and we then tabulated our results on the board.

$$F_1 + F_2 = F_4 - 1; \quad F_1 + F_2 + F_3 = F_5 - 1; \cdots F_1 + F_2 + \cdots + F_n = F_{n+2} - 1 .$$

This was another attempt to have the students use notation to express the patterns rather than just verbalizing the result in English. Earlier I had suggested that each student should use a notebook to write down the results of the previous work and I requested that they write down what we had done on the board. Although many of the students felt uncomfortable with the notation and indicated that they did not like using F_n, they understood that the notation said the same as "the sum of the first n Fibonacci numbers can be found by going two more Fibonacci numbers and subtracting 1."

After the students had worked with the pattern for finding the sum of the Fibonacci numbers, I asked them to find

$$F_1 + F_2 + F_3 + \cdots + F_{25}$$

in sixty seconds for a brief quiz. It is interesting that approximately one-third of the class was not able to connect the problem to our previous work; one-fourth of the class found the result correctly; and the remainder of the class used the right idea in trying to find the solution but could not remember which Fibonacci number they should get even though they knew that if they added one to the result they would get a Fibonacci number. However, considering the make-up of the class, I was very encouraged.

We next worked on

$$F_2 + F_4 + F_6 + \cdots + F_{2n} .$$

I placed the problem on the board in the above form and was pleasantly surprised to find a general acceptance of the notation at this point. There was a little concern over the expression 2n and we spent some time answering the question of the value of 2n for n = 1, n = 2, n = 3, etc. The students indicated they understood; however, as we went on, I had to continually remind the students of this form for the even index in the Fibonacci code.

We then went on to

$$F_1 + F_3 + F_5 + \cdots + F_{2n-1} .$$

Again there was concern over 2n - 1, and we had another chance to discuss an algebraic expression. This was a second opportunity to introduce the concept of the variable without making the process a painful experience.

Interest at this point was running high and I felt that the class was sharing my enthusiasm. Even those who usually were apathetic to any of the material presented during the first semester were becoming involved.

I then presented

$$F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 .$$

At this point there was a little negative reaction that this problem was too hard; some of the students indicated that this problem should be in an algebra class rather than Math I. Since we had discussed scientific notation earlier in the year working with googols and googolplexes, I reviewed what the exponent 2 represented in each term of the series.

The presentation became a little more detailed this time and I found a great number of the students independently making out a table of squares of the Fibonacci numbers to help them find the pattern. I was very enthusiastic over the idea that some of the students were voluntarily doing more than was required. The pattern was finally established but not until we had a chance to discuss what was meant by a factor. In particular

$$F_1^2 + F_2^2 + F_3^2 = 6$$

was an excellent opportunity to both illustrate that they should use the sum of several terms before trying to establish the pattern and to see that the factors would be Fibonacci numbers if they were not the trivial set of 1 and 6. The results of the ensuing discussion were too extensive to record here, but let it suffice to say that we discussed division, multiplication, exponents, factoring, prime numbers, addition, and general notation without appearing to meet any negative reaction.

I was very pleased with our progress but felt that if I were to maintain the existing level of enthusiasm I would have to try to vary the class activity more than I was doing via the presentations on the board and individual work. As such, I decided to have the class work in groups.

Thus, at the end of the second week I told the class that we would start working in groups the following week. I asked seven students, who I felt could act as group leaders, to make three lists of four students each for their possible groups. I indicated I would try to form their groups from these lists. With this done, I reviewed what had been accomplished during the first two weeks.

On Monday I indicated who was to be in each group and handed out two dittos. One summarized our work up till then; and the second was a set of 15 problems which would be done in their group work, Table 1.

Table 1

1. Find the sum of the Lucas numbers.
2. Find the sum of $L_1 + L_2 + L_5 + L_7 + L_9 + \cdots$.
3. Determine the sum of $L_2 + L_4 + L_6 + L_8 + \cdots$.
4. What is the sum of the squares of the Lucas numbers?
5. Form $F_1^2 + F_2^2$; $F_2^2 + F_3^2$; $F_3^2 + F_4^2$; etc.
6. Do the same as in No. 5 for the Lucas numbers.
7. Find the sum of $L_1 + L_5 + L_9 + L_{13} + L_{17} \cdots$.
8. Determine the sum of $L_2 + L_6 + L_{10} + L_{14} \cdots$.
9. Find the sum of $L_3 + L_7 + L_{11} + L_{15} + L_{19} \cdots$.

10. Determi Table 1 (Continued)

10. Determine the sum of $L_4 + L_8 + L_{12} + L_{16} + L_{20} \cdots$.
11. Find a pattern which works for Nos. 7, 8, 9, and 10.
12. Find F_1L_2 and F_2L_1 and their difference; find F_2L_2 and F_3L_2 and their difference; F_3L_4 and F_4L_2 and their difference; etc.
13. Find F_1L_3 and F_3L_1 and their difference; continue as in No. 12.
14. Find F_1L_4 and F_4L_1 and their difference; continue as in No. 13.
15. The process begun in Nos. 12, 13, and 14 can be continued to spacings of three, four, five, etc. Can you find a pattern in the answers?

Each group was to consist of the leader and three students to work with the leader in the group. I placed the seven groups in clusters about the room with the following directions:

(1) If a member of the group had a question, he was to ask the group leader.

(2) The group leader would discuss problems with me so as to explain the problems to the group.

(3) The individuals in the group, excluding the leader, would receive their grade based on their work in the group, their notebooks, and an oral test.

(4) The group leaders would receive their grades based on their understanding of the material discussed, and more importantly on how much knowledge they could impart to each student in their group, i.e., their grades rested on how much their group knew.

Before actually placing them in groups we discussed a second sequence, the Lucas sequence. For homework they were to find the first twenty-five terms of the sequence starting with 1, 3, 4, etc. We also discussed the notation $L_1, L_2, L_3, \cdots$ for the Lucas numbers.

Although the class was homogenous in that it was basically low ability, there was enough diversity in ability so that the leaders were sufficiently advanced in the material to meet their obligations. I believe that much of the success we had in the group work was based on the selection of the groups and the ability of the seven group leaders.

For the next three weeks I worked with the group leaders and the groups themselves encouraging, explaining, and making sure that each group got the help it needed. I tried to work with each leader and group at least once each day, but I was pleasantly surprised with the way in which the groups maintained their enthusiasm and worked in as mature a fashion as could be desired when I was busy with other groups. Quite often I worked with entire groups discussine one problem. In many cases the group leaders displayed remarkable behavior in directing the research and explaining to the students in their groups a particular pattern. I was particularly impressed with the patience and understanding displayed by the group leaders.

At the end of the first week in groups, I asked each group leader to submit a progress report on his group. Following is an example of the type of response I received.

> "Our group has progressed fairly with one exception, * * *. Now I see how hard it is for a teacher to try to teach her something. She just won't even try to learn, and when I tell her to try, she sais I can't. Sometimes she catches on, but after she gets to a part that is too hard for her (she thinks) she quits and talks or else just plain forgets it. I don't know what I'll do if she won't learn. • • •"

Each student in the group was to keep a record of his work and the patterns discovered were to be listed. I collected these notebooks at the end of each week and was extremely pleased with the results. I wish it were possible to include one of the notebooks in the article. Again let it suffice to say that the notebooks do justice to those collected from students in a freshman algebra course.

During the third week in groups I had each student go before the class using the overhead projector. They were to present the solution to one of the problems from the work done during the five weeks. This problem was given to them when they went up to the overhead projector. They were allowed to take their lists of Fibonacci and Lucas numbers with them to the projector.

The presentations were of fine quality with the students explaining how they were establishing the patterns. I might add that the patterns were not memorized but rediscovered while working in front of the class with the overhead projector. I tried to present a problem geared to the ability of each student and I started the presentations with students who would present their patterns in a good style to serve as examples for the other students. During many of the presentations it was necessary to make suggestions via questions as to what should be done. I made every attempt to make sure that enough help was given so that the student eventually was able to find the pattern. Grades for the oral presentations were based on how well the presentation was delivered and the amount of help given to the student. During the presentations, the other students were asked to try the pattern at their desks. It was fascinating to see the involvement of the class in some cases as the person giving the presentation would struggle with a pattern.

Generally speaking, the oral presentations were a highlight of the five weeks. I am sure the class approached the presentations with some less enthusiasm, but the cooperation was very satisfying from the majority of the students.

With the conclusion of the unit, grades were given out to each student if such were requested. The student reaction to the work on Fibonacci numbers was very positive. One girl even went so far as to say that the material should be part of the required curriculum for the ninth grade.

The results of the five week unit on Fibonacci numbers were very encouraging. The change in student attitude, one of the three goals, was readily observed. Since the unit was presented later in the year, there was no opportunity to observe whether the changes in attitude observed would have transferred to work presented earlier in the year.

The material presented following the unit involved working with areas, volumes, perimeters, and circumferences of basic geometric figures. Again the material was new to the class and the student reaction was one of acceptance and willingness to work on the problems. Although there was not a great deal of enthusiasm present, the reaction of the class was satisfying, considering we were in the last six weeks of the school year.

I am looking forward to expanding the Fibonacci unit for next year with three classes to include work on phyllotaxis and geometric relationships. Also, I hope to present the unit earlier in the year to explore more fully the transfer of changes of attitude toward mathematics in general.

* * * * *