BASES FOR INTERVALS OF REAL NUMBERS

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1. INTRODUCTION.

In this paper we discuss the problem of representing uniquely each real number in the interval $(0, c]$, where c is any positive real number, as an infinite series of terms selected from a sequence (b_n) of real numbers. We choose an integer $k \geq 1$ and require that any two terms of (b_n) whose suffices differ by less than k shall not both be used in the representation of any given real number. The precise definitions and results are given in the next section.

In an earlier paper [2] we discussed an analogous problem of representing the integers in arbitrary infinite intervals.

2. STATEMENT OF RESULTS

Throughout this paper $k \geq 1$ is an integer. Also the subscript of the initial term of any sequence is the number 1; e.g., $(c_n) = (c_1, c_2, \cdots)$.

In order to prove our main result, which is theorem 2, we need a result which we give in a slightly generalized form as Theorem 1. Let (c_n) be a sequence of positive real numbers which obey the linear recurrence relation

$$a_1 c_{n+k} + a_2 c_{n+k-1} + \cdots + a_k c_{n+1} - c_n = 0 \tag{2.1}$$

for $n \geq 1$, where $a_1, \cdots, a_k$ are non-negative real numbers independent of n, and $a_1 \geq 0$. The auxiliary polynomial $g(z)$ of this recurrence relation is given by

$$g(z) = a_1 z^k + a_2 z^{k-1} + \cdots + a_k z - 1 .$$

It is clear that $g(z)$ has just one positive real root ρ, and that this root is simple.

(Received March 1965)

Theorem 1. If the sequence (c_n) is strictly decreasing, and ρ is smaller than the modulus of any other root of $g(z)$, then $\rho < 1$ and $c_n = A\rho^n$ for $n \geq 1$, where A is a positive real constant.

We now define a k-series base for the interval of real numbers $(0, c]$, where c is any positive real constant. This is analogous to the concept of an (h, k) base for the set of integers as an interval; this concept was given in the earlier paper [2].

Definition. A sequence (b_n) of real numbers is a k-series base for $(0, c]$ if each real number $r \in (0, c]$ has a unique representation

$$r = b_{i_1} + b_{i_2} + \cdots , \tag{2.2}$$

where

$$i_{\nu+1} \geq i_\nu + k$$

for $\nu \geq 1$, and further, every such series converges to a sum $r \in (0, c]$.

It is clear that the polynomial $f(z) = z^k + z - 1$ has just one positive real root θ, that θ is a simple root, and that $\theta < 1$. Let R be a real number. We now enunciate our main result.

Theorem 2. Let (b_n) be a sequence of real numbers such that $b_n \geq b_{n+1} > 0$ for $n \geq 1$. Then (b_n) is a k-series base for $(0, \theta^R]$ if and only if

$$b_n = \theta^{R+n}$$

for $n \geq 1$.

It is not true that all k-series bases are decreasing. For instance, when $k = 2$, the series $(1, 2, \theta, \theta^2, \cdots)$ is a k-series base for $(0, 2 + \theta]$. However, A. Oppenheim has shown that if the sequence (b_n) is a k-series base for $(0, c]$ for some $c > 0$, and if N is an integer such that $b_n \geq b_{n+1} > 0$ for $n \geq N$ then $b_n = A\theta^n$ for $n > k$, where A is some positive constant. It is not known if all k-series bases (for $k \geq 2$) are ultimately decreasing.

It follows from Theorem 2 that the sequence

$$(\theta^{-N+1}, \theta^{-N+2}, \cdots, \theta^{-1}, \theta^0, \theta^1, \cdots) \tag{2.3}$$

is a k-series base for $(0, \theta^{-N}]$, where N is any positive integer. Hence if r is any positive real number, and L and M are positive integers such that both $\theta^{-L} > r$ and $\theta^{-M} > r$, then the k-series representation of r in terms of the sequence (2.3) with $N = L$ is the same as with $N = M$. For shortness, therefore, we can refer to this as the 'θ-representation' of r. Then, if an initial minus sign is used in representing negative numbers, we can give a unique 'θ-representation' for any real number. A 'θ-representation' of real numbers is akin to decimal representation, but is much more closely related to binary representation since when $k = 1$ the 'θ-representation' and the binary representation of the same real number are the same (for when $k = 1$, $\theta = \frac{1}{2}$).

A further observation is that any sum T of a finite number of terms of the sequence $(\theta^{R+1}, \theta^{R+2}, \cdots)$, where R is any real number, in the form

$$T = \theta^{i_1} + \theta^{i_2} + \cdots + \theta^{i_{\alpha-1}} + \theta^{i_\alpha - 1},$$

where $i_{\nu+1} \geq i_\nu + k$ for $1 \leq \nu < \alpha$, can be written in the form

$$T = \sum_{\nu=1}^{\infty} \theta^{i_\nu},$$

where $i_{\nu+1} \geq i_\nu + k$ for $\nu \geq 1$, simply by putting

$$\theta^{i_\alpha - 1} = \sum_{\nu=1}^{\infty} \theta^{i_\alpha + \nu k}, \tag{2.4}$$

(The relation (2.4) follows from the relations

$$\sum_{\nu=0}^{\infty} \theta^{\nu} < \infty,$$

and

$$\theta^{i_\alpha - 1 + nk} = \theta^{i_\alpha + nk} + \theta^{i_\alpha - 1 + (n+1)k}$$

for $n \geq 0$, both of which are very easily proved.) This fact is analogous to the decimal equation $1 = 0.\dot{9}$ or the binary equation $1 = 0.\dot{1}$.

3. PROOF OF THEOREM 1

We first prove Lemma 1, an equivalent form of which occurred originally in [3] and was also quoted in [4].

Lemma 1. If $\alpha_1, \alpha_2, \cdots, \alpha_p$ are real numbers then there exists an increasing sequence (n_j) of positive integers such that

$$\exp(in_j\alpha_1) \to 1, \quad \exp(in_j\alpha_2) \to 1, \quad \cdots \exp(in_j\alpha_p) \to 1 \text{ as } j \to \infty.$$

Proof. For x a real number, let $\bar{x}$ be the number differing from x by a multiple of 2π such that $-\pi < \bar{x} \leq \pi$. We prove the lemma by showing that if we are given any positive real number $\epsilon > 0$, and any positive integer N, then we find an integer $n \geq N$ such that

$$\left|\overline{n\alpha_s}\right| < \epsilon \quad \text{for} \quad 1 \leq s \leq p.$$

Let M be the region in p-dimensional space in which each coordinate ranges from $-\pi$ to π. Let the range of each coordinate be divided into m equal parts, where

$$m > \frac{2\pi}{\epsilon}$$

is an integer. Then M is divided into m^p equal parts. Consider now the $m^p + 1$ points

$$(\overline{N\nu\alpha_1}, \overline{N\nu\alpha_2}, \cdots, \overline{N\nu\alpha_p}) \text{ for } 1 \leq \nu \leq m^p.$$

One part of M must contain two of these points; let the corresponding indices be ν_1 and ν_2. Then clearly

$$\left|\overline{N(\nu_1 - \nu_2)\alpha_s}\right| < \frac{2\pi}{m} < \epsilon$$

for $1 \le s \le p$, and

$$\left| \nu_1 - \nu_2 \right| \ge 1 \ .$$

We put

$$n = N\left|\nu_1 - \nu_2\right| \ ;$$

this proves Lemma 1.

Since (c_n) obeys the recurrence relation (2.1), c_n can be expressed in the form

$$c_n = \sum_{s=1}^{u} \left(\sum_{t=0}^{v_s} n^t B_{st} \right) \xi_s^n \quad \text{for } n \ge 1 \ , \tag{3.1}$$

where the numbers ξ_s are the distinct roots of $g(z)$, the number $(v_s + 1)$ is the multiplicity of the root ξ_s for $1 \le s \le u$, and the numbers B_{st} are suitable complex constants. Let $= \xi_{s'}$. We consider two cases.

Case 1. $B_{st} = 0$ when $(s, t) \neq (s', 0)$. Then by (3.1),

$$c_n = B_{s'0}\rho^n \quad \text{for} \quad n \ge 1 \ . \tag{3.2}$$

Since c_1, $\rho > 0$ it follows by (3.2) that

$$B_{s'0} = \frac{c_1}{\rho} \ ,$$

a positive constant. Since (c_n) is a decreasing sequence, $\rho < 1$. Hence the theorem is true in this case.

Case 2. $B_{st} \neq 0$ for at least one pair $(s, t) \neq (s', 0)$. This implies that $k \ge 2$. We shall deduce a contradiction. By rearranging the terms in (3.1) if necessary, there is a number p where $1 \le p \le u$, and a number q, where $0 \le q \le \min (v_1, v_2, \cdots, v_p)$ such that

(i) For $1 \le s \le p$, $B_{sq} \ne 0$, $|\xi_s| = |\xi_1|$ and $B_{st} = 0$ for $q \le t \le v_s$,

(ii) for $p < s \le u$, if $|\xi_s| = |\xi_1|$ then $B_{st} = 0$ for $q < t \le v_s$, and if $|\xi_s| > |\xi_1|$ then $B_{st} = 0$ for $0 \le t \le v_s$.

Then by (3.1)

$$c_n = \sum_{s=1}^{p} B_{sq} n^q \xi_s^n + R , \tag{3.3}$$

where R is the sum of a finite number of non-zero terms of the form $Cn^{\gamma}\xi_{\delta}^{n}$, where C is a complex constant and either $|\xi_\delta| = |\xi_1|$ and $\gamma < q$, or $|\xi_\delta| < |\xi_1|$ Our assumption implies that either

$$|\xi_1| > \rho \quad \text{or} \quad q > 0 . \tag{3.4}$$

If $|\xi_\delta| < |\xi_1|$ then $n^{\gamma}|\xi_\delta|^n/|\xi_1|^n \to 0$ as $n \to \infty$. Hence

$$R/|\xi_1|^n n^q \to 0 \quad \text{as} \quad n \to \infty . \tag{3.5}$$

For $1 \le s \le p$, let $\xi_s = r_s \exp(i\alpha_s)$, where r_s and α_s are the modulus and argument of ξ_s respectively. Then by (3.5) and (3.4) respectively,

$$R/r_1^n n^q \to 0 \quad \text{as} \quad n \to \infty \tag{3.6}$$

and either

$$r_1 > \rho \quad \text{or} \quad q > 0 . \tag{3.7}$$

Further, let w be the smallest positive integer such that when $n = w$, $E_n \ne 0$, where

$$E_n = \sum_{s=1}^{p} B_{sq} \exp(in\alpha_s), \quad \text{for} \quad n \ge 1 .$$

The number w exists, for otherwise $B_{sq} = 0$ for $1 \leq s \leq p$. From (3.3) and (3.6)

$$\frac{c_{n_j}}{r_1^n n^q} = \sum_{s=1}^{p} B_{sq} \exp(iw\alpha_s) \cdot \exp(i(n-w)\alpha_s) + 0(1) \quad \text{as} \quad n \to \infty.$$

By Lemma 1 there exists a sequence (n_j) of positive integers such that

$$\frac{c_{n_j}}{r_1^{n_j} n_j^q} = E_w + 0(1) \quad \text{as} \quad j \to \infty. \tag{3.8}$$

It is clear from (3.8) that E_w is real and positive; since (c_n) is a decreasing sequence, we have also that $r_1 < 1$ and hence $\rho < 1$.

By (3.7) and (3.8) there exists an integer m such that

$$\frac{c_m}{\rho^m} = \frac{c_m}{r_1^m m^q} \cdot \left(\frac{r_1}{\rho}\right)^m \cdot m^q > \left(\frac{c_1}{\rho}\right) \rho^{1-k} .$$

Hence,

$$c_{m-k+1} > c_{m-k+2} > \cdots > c_m > \left(\frac{c_1}{\rho}\right) \rho^{m-k+1} > \left(\frac{c_1}{\rho}\right) \rho^{m-k+2} > \cdots > \left(\frac{c_1}{\rho}\right) \rho^m . \tag{3.9}$$

Therefore,

$$\begin{aligned} c_{m-k} = a_1 c_m + a_2 c_{m-1} + \cdots + a_k c_{m-k+1} &> \left(\frac{c_1}{\rho}\right) (a_1 \rho^m + a_2 \rho^{m-1} + \\ + \cdots + a_k \rho^{m-k+1}) &= \left(\frac{c_1}{\rho}\right) \rho^{m-k} . \end{aligned} \tag{3.10}$$

Using (3.9) and (3.10) we find in a similar way that

$$c_{m-k-1} > \left(\frac{c_1}{\rho}\right)\rho^{m-k-1}$$

and

$$c_{m-k-2} > \left(\frac{c_1}{\rho}\right)\rho^{m-k-2}$$
$$\vdots$$

and so on, until

$$c_1 > \left(\frac{c_1}{\rho}\right)\rho ,$$

a contradiction. Hence Case 2 does not occur. This proves Theorem 1.

4. PROOF OF THEOREM 2

The sequence (b_n) is clearly a k-series base for $(0, \theta^R]$ if and only if

$$\left(\frac{b_n}{\theta^R}\right)$$

is a k-series base for $(0, 1]$. Hence without loss of generality we assume that $R = 0$, so that we shall be discussing k-series bases for $(0, 1]$.

Lemma 2.

$$\theta^n = \sum_{\nu=0}^{\infty} \theta^{n+1+\nu k} \qquad \text{for} \quad n \geq 0 .$$

Proof. Since θ is a root of $f(z) = z^k + s - 1$ and $0 < \theta < 1$, we see that

$$\sum_{\nu=0}^{m} \theta^{n+1+\nu k} = \theta^{n+1} \sum_{\nu=0}^{\infty} (\theta^k)^\nu = \theta^{n+1}\left(\frac{1}{1-\theta^k}\right) = \frac{\theta^{n+1}}{\theta} = \theta^n .$$

for $m \geq 0$. Since $\theta < 1$ it follows that

$$\sum_{\nu=0}^{\infty} \theta^{1+\nu k} = 1 ,$$

and hence

$$\theta^n = \sum_{\nu=0}^{\infty} \theta^{n+1+\nu k} \quad \text{for} \quad n \geq 0 ,$$

as required.

Proof of sufficiency. We show that (θ^n) is a k-series base for $(0, 1]$.

Let $0 < x \leq 1$. First we construct inductively a sequence (i_ν) of positive integers such that $i_{\nu+1} \geq i_\nu + k$ for $\nu \geq 1$, and

$$\theta^{i_m - 1+k} \geq x - \sum_{\nu=1}^{m} \theta^{i_\nu} > 0 , \tag{4.1}$$

for $m \geq 1$. The integer i_1 is chosen so that

$$\theta^{i_1 - 1} \geq x > \theta^{i_1} ,$$

and since $\theta + \theta^k = 1$ we see that

$$\theta^{i_1-1+k} = \theta^{i_1-1} - \theta^{i_1} \geq x - \theta^{i_1} > 0 .$$

Let $t \ge 1$ be an integer and suppose that $i_1, i_2, \cdots, i_t$ are chosen so that (4.1) holds for $m = t$, and $i_{\nu+1} \ge i_\nu + k$ for $1 \le \nu < t$. Then we choose i_{t+1} such that

$$\theta^{i_{t+1}-1} \ge x - \sum_{\nu=1}^{t} \theta^{i_\nu} > \theta^{i_{t+1}} \quad . \tag{4.2}$$

Hence

$$\theta^{i_{t+1}-1+k} = \theta^{i_{t+1}-1} - \theta^{i_{t+1}} \ge x - \sum_{\nu=1}^{t+1} \theta^{i_\nu} > 0 \ .$$

From (4.2) and the assumption that (4.1) holds for $m = t$ it follows that

$$\theta^{i_t-1+k} \ge \theta^{i_{t+1}-1} \ .$$

Hence $i_{t+1} \ge i_t + k$. The construction of the sequence (i_ν) follows by induction.

Since $\theta < 1$ it follows from (4.1) that there exists a representation of x in the form

$$x = \sum_{\nu=1}^{\infty} \theta^{i_\nu} \ , \tag{4.3}$$

where $i_1 \ge 1$ and $i_{\nu+1} \ge i_\nu + k$ for $\nu \ge 1$.

This representation of x is unique. For otherwise we may assume without loss of generality that

$$\sum_{\nu=1}^{\infty} \theta^{i_\nu} = \sum_{\nu=1}^{\infty} \theta^{j_\nu} \ ,$$

where $i_1 \geq 1$ and $i_{\nu+1} \geq i_\nu + k$ for $\nu \geq 1$, $j_1 \geq 1$ and $j_{\nu+1} \geq j_\nu + k$ for $\nu \geq 1$, and $i_1 < j_1$. Then

$$\theta^{i_1} < \sum_{\nu=1}^{\infty} \theta^{i_\nu} = \sum_{\nu=1}^{\infty} \theta^{j_\nu} \leq \sum_{\nu=0}^{\infty} \theta^{j_1+\nu k} = \theta^{j_1-1}$$

by Lemma 2. Hence $i_1 > j_1 - 1$, which contradicts the assumption that $i_1 < j_1$.

Since $\theta > 0$, no non-positive numbers can be represented in the form (4.3). By Lemma 2,

$$\sum_{\nu=0}^{\infty} \theta^{1+\nu k} = 1$$

and so 1 is the largest number which has a representation in the form (4.3). Hence (θ^n) is a k-series base for $(0, 1]$. This completes the proof of the sufficiency.

Proof of necessity. We show that if the sequence (b_n) is a k-series base for $(0, 1]$, and if $b_{n+1} \geq b_n > 0$ for $n \geq 1$, then $b_n = \theta^n$ for $n \geq 1$.

For shortness we write $b_0 = 1$, but as stated earlier, by the sequence (b_n) we mean the sequence $(b_1, b_2, \cdots)$. The sequence (b_n) is strictly decreasing, for if $b_i = b_j$ for $i \neq j$ then clearly some numbers have more than one k-series representation. For $n \geq 1$ we define

$$B_n = \left\{ r \mid r = \sum_{\nu=1}^{\infty} b_{i_\nu}\ ;\ i_1 = n,\quad i_{\nu+1} \geq i_\nu + k \text{ for } \nu \geq 1 \right\}.$$

We denote by $\overline{B}_n$ the least upper bound of B_n. Since (b_n) is a positive strictly decreasing sequence it follows that

$$\overline{B}_n = \sum_{\nu=0}^{\infty} b_{n+\nu k} \quad \text{for} \quad n \geq 1 \tag{4.4}$$

and

$$\overline{B}_n > \overline{B}_{n+1} > 0 \quad \text{for} \quad n \geq 1. \tag{4.5}$$

It follows from (4.5) that there exists a non-negative real number ℓ such that $\overline{B}_n \to \ell$ as $n \to \infty$. But, by (4.4)

$$\sum_{\nu=0}^{\infty} b_{1+\nu k} = \lim_{m \to \infty}\left(\sum_{\nu=0}^{m} b_{1+\nu k} + \overline{B}_{1+(m+1)k}\right) = \sum_{\nu=0}^{\infty} b_{1+\nu k} + \ell,$$

so that $\ell = 0$. Hence

$$\overline{B}_n \to 0 \quad \text{as} \quad n \to \infty\,. \tag{4.6}$$

We now prove by induction upon n that

$$\overline{B}_{n+1} = b_n \tag{4.7}$$

for $n \geq 0$. Since (b_n) is a k-series base for $(0,1]$ it follows from (4.5) that $\overline{B}_1 = 1$, and so (4.7) is true when $n = 0$. Let $m \geq 1$ be a positive integer and suppose as an induction hypothesis that (4.7) is true for $0 \leq n < m$.

If $b_m > \overline{B}_{m+1}$ then there is no k-series representation for $\frac{1}{2}(b_m + \overline{B}_{m+1})$. Suppose that $b_m < \overline{B}_{m+1}$. Then we can construct inductively a sequence (j_ν) of positive integers, where $j_1 = m$ and $j_{\nu+1} \geq j_\nu + k$ for $\nu \geq 1$, such that for $\nu \geq 1$ there are infinitely many positive integers n satisfying

$$\overline{B}_{m+1} + \frac{1}{n} \in B_j \quad \text{if } \nu = 1\ ,$$

or

$$\overline{B}_{m+1} - b_{j_1} - b_{j_2} - \cdots - b_{j_{-1}} + \frac{1}{n} \in B_j \quad \text{if } \nu \geq 2\,. \tag{4.8}$$

By (4.5) and the induction hypothesis,

$$b_0 = \overline{B}_1 > b_1 = \overline{B}_2 > \cdots > b_{m-1} = \overline{B}_m > \overline{B}_{m+1},$$

and so there are infinitely many positive integers n such that

$$\overline{B}_{m+1} + \frac{1}{n} \in B_m .$$

Let $\delta > 1$ be an integer and suppose that the first $\delta - 1$ terms of (j_ν) are chosen. Then for infinitely many positive integers n,

$$\begin{cases} \overline{B}_{m+1} + \frac{1}{n} \in B_{j_{\delta-1}}, & \text{if } \delta = 2 \\ \overline{B}_{m+1} - b_{j_1} - b_{j_2} - \cdots - b_{j_{\delta-2}} + \frac{1}{n} \in B_{j_{\delta-1}}, & \text{if } \delta \geq 2 . \end{cases}$$

Hence

$$\overline{B}_{m+1} - b_{j_1} - b_{j_2} - \cdots - b_{j_{\delta-1}} + \frac{1}{n} \in \bigcup_{i=j_{\delta-1}+k}^{\infty} B_i$$

for infinitely many positive integers n. Therefore

$$\overline{B}_{m+1} - b_{j_1} - b_{j_2} - \cdots - b_{j_{\delta-1}} \geq 0 .$$

However, if $\overline{B}_{m+1} = b_{j_1} + b_{j_2} + \cdots + b_{j_{\delta-1}}$, then, by replacing $b_{j_{\delta-1}}$ by its k-series representation we obtain a k-series representation for $\overline{B}_{m+1}$ different from the k-series representation given in (4.4), and this contradicts the fact that (b_n) is a k-series base. Therefore by (4.6) there exists a positive integer q such that

$$B_{m+1} - b_{j_1} - b_{j_2} - \cdots - b_{j_{\delta-1}} > \overline{B}_q .$$

Hence

$$\overline{B}_{m+1} - b_{j_1} - b_{j_2} - \cdots - b_{j_{\delta-1}} + \frac{1}{n} \in \bigcup_{i=j_{\delta-1}+k}^{q} B_i$$

for infinitely many positive integers n. Hence there exists $j_\delta \geq j_{\delta-1} + k$ such that

$$\overline{B}_{m+1} - b_{j_1} - b_{j_2} - \cdots - b_{j_{\delta-1}} + \frac{1}{n} \in B_{j_\delta}$$

for infinitely many positive integers n. The construction of the sequence (j_ν) follows by induction.

We deduce from (4.8) that for $\nu > 1$,

$$0 < \overline{B}_{m+1} - b_{j_1} - b_{j_2} - \cdots - b_{j_{\nu-1}} \leq B_{j_\nu}$$

and by (4.6) it follows that

$$\overline{B}_{m+1} = \sum_{\nu=1}^{\infty} b_{j_\nu} .$$

This k-series representation for $\overline{B}_{m+1}$ is different from that given in (4.4), which contradicts the fact that (b_n) is a k-series base. Hence $\overline{B}_{m+1} = b_m$, and it follows by induction that (4.7) holds for all $n \geq 0$.

By (4.4), for $n \geq 0$,

$$\overline{B}_{n+1} = \sum_{\nu=0}^{\infty} b_{n+1+\nu k} = b_{n+1} + \sum_{\nu=0}^{\infty} b_{n+k+1+\nu k} = b_{n+1} + \overline{B}_{n+k+1} ,$$

and therefore, by (4.7)

$$b_n = b_{n+1} + b_{n+k} \quad \text{for} \quad n \geq 0 .$$

The number θ is the positive real root of the auxiliary polynomial $f(z) = z^k + z - 1$ of this recurrence relation. The modulus of any other root of $f(z)$ is greater than θ. For if $|z| < \theta$, then since $\theta < 1$,

$$|f(z)| = |1 - z(1 + z^{k-1})| \geq 1 - |z|(1 + |z|^{k-1}) > 1 - \theta(1 + \theta^{k-1}) = 0 ,$$

whilst if $f(z) = 0$ and $|z| = \theta$, then

$$1 - |z| - |1 - z| = 1 - |z| - |f(z) - z + 1| = 1 - |z| - |z|^k = 0 ,$$

so that

$$|1 - z| = 1 - |z| ,$$

and hence $\arg z = 0$ so that $z = \theta$.

By Theorem 1, therefore, for some positive constant A, $b_n = A\theta^n$ for $n \geq 1$. However, we have shown that (θ^n) is a k-series base for $(0,1]$, and so it follows that $A = 1$. This completes the proof of the necessity and of Theorem 2.

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The Editor acknowledges with gratitude the receipt of $350 from the Academic Vice President of San Jose State College toward our publication expenses.

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