ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-148 Proposed by James E. Desmond, Florida State University, Tallahassee, Florida

Prove or disprove: There exists a positive integer m such that

$$m \text{ times } \left\{ F_{F_{\cdot_{\cdot_{\cdot_{F_{F_n}}}}}} \right.$$

is composite for all integers $n > 5$.

H-149 Proposed by Charles R. Wall, University of Tennessee, Knoxville, Tenn.

For $s = \sigma + it$ let

$$P(s) = \sum p^{-s} ,$$

where the summation is over the primes. Set

$$\sum_{n=1}^{\infty} a(n)n^{-s} = [1 + P(s)]^{-1} ,$$

$$\sum_{n=1}^{\infty} b(n)n^{-s} = [1 - P(s)]^{-1} .$$

Determine the coefficients a(n) and b(n).

H-150 Proposed by M. N. S. Swamy, Nova Scotia Technical College, Halifax, Canada

Show that

$$25\sum_{p=1}^{n-1}\sum_{q=1}^{p}\sum_{r=1}^{q} F_{2r-1}^2 = F_{4n} + (n/3)(5n^2 - 14) ,$$

where F_n is the n^{th} Fibonacci number.

H-151 Proposed by L. Carlitz, Duke University, Durham, N. Carolina

A. Put

$$(1 - ax^2 - bxy - cy^2)^{-1} = \sum_{m,n=0}^{\infty} A_{m,n}x^m y^n .$$

Show that

$$\sum_{n=0}^{\infty} A_{n,n}x^n = \left\{1 - 2bx + (b^2 - 4ac)x^2\right\}^{-\frac{1}{2}} .$$

B. Put

$$(1 - ax - bxy - cy)^{-1} = \sum_{m,n=0}^{\infty} B_{m,n} x^m y^n .$$

Show that

$$\sum_{n=0}^{\infty} B_{n,n} x^n = \left\{ (1 - bx)^2 - 4acx \right\}^{-\frac{1}{2}} .$$

H-152 Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Let m denote a positive integer and F_n the n^{th} Fibonacci number. Further let $\{c_k\}_{k=1}^{\infty}$ be the sequence defined by

$$\{c_k\}_{k=1} \equiv \{ \underbrace{F_n^m, F_n^m, \cdots, F_n^m}_{2^{m-1} \text{ copies}} \}_{n=1}^{\infty}$$

Prove that $\{c_k\}_{k=1}^{\infty}$ is complete; i.e., show that every positive integer, n, has at least one representation of the form

$$n = \sum_{k=1}^{p} \alpha_k c_k ,$$

where p is a positive integer and

$$\alpha_i = 0 \text{ or } 1 \text{ if } k = 1, 2, \cdots, p - 1$$

$$\alpha_p = 1$$

C.f. V. E. Hoggatt, Jr., and C. King, Problem E1424, American Mathematical Monthly, Vol. 67 (1960), p. 593 and J. L. Brown, Jr., "Note on Complete Sequences of Integers," American Mathematical Monthly, Vol. 67 (1960), pp. 557-560.

SOLUTIONS

POWER PLAY

H-109 Proposed by George Ledin, Jr., San Francisco, Calif.

Solve

$$x^2 + y^2 + 1 = 3xy$$

for all integral solutions and consequently derive the identity

$$F^2_{6k+7} + F^2_{6k+5} + 1 = 3F_{6k+7}F_{6k+5} .$$

Solution by H. V. Krishna, Manipal Engineering College, Manipal, India

Let the equation in question be expressed as

(1) $$(x - 3y/2)^2 - 5(y/2)^2 = -1 .$$

The general solution of (1) is therefore given by

(2) $$x - (3y/2) = \tfrac{1}{2}\left\{(p + \sqrt{5}q)^{2n-1} + (p - \sqrt{5}q)^{2n-1}\right\}$$
$$(y/2) = 1/(2\sqrt{5})\left\{(p + \sqrt{5}q)^{2n-1} - (p - \sqrt{5}q)^{2n-1}\right\}$$

where (p,q) is a particular solution of (1).

Hence (2) reduces to $y = F_{2n-1}$ and $x = (1/2)(L_{2n-1} + 3F_{2n-1})$ for $p = \frac{1}{2}$ and $q = \frac{1}{2}$.

On using $L_{2n-1} + F_{2n-1} = 2F_{2n}$,

$$x = \tfrac{1}{2}\left\{2(F_{2n} + F_{2n-1})\right\} = F_{2n+1} ,$$

whence the desired identity follows for $n = 3(k + 1)$.

Also Solved by A. Shannon.

TRIG OR TREAT

H-111 Proposed by John L. Brown, Jr., Pennsylvania State University, State College, Pa.

Show that

$$L_n = \prod_{k=1}^{[n/2]} \left\{ 1 + 4\cos^2 \frac{2k-1}{n}\left(\frac{\pi}{2}\right) \right\} \text{ for } n \geq 1 .$$

Solution by M. N. S. Swamy, Nova Scotia Technical College, Halifax, Canada.

We know from the solution of Problem H-64 (Fibonacci Quarterly, Vol. 5, Feb. 1967, p. 75), that

$$L_n = \prod_{j=1}^{n} \left\{ 1 - 2i \cos \frac{(2j-1)\pi}{2n} \right\}, \quad i = \sqrt{-1} .$$

If n is odd, then

$$L_{2n+1} = \prod_{j=1}^{2n+1} \left\{ 1 - 2i \cos \frac{(2j-1)\pi}{2(2n+1)} \right\}$$

$$= \prod_{j=1}^{n} \left\{ 1 - 2i \cos \frac{(2j-1)\pi}{2(2n+1)} \right\} \cdot \prod_{k=n+2}^{2n+1} \left\{ 1 - 2i \cos \frac{(2k-1)\pi}{2(2n+1)} \right\}$$

$$\cdot \left\{ 1 - 2\cos \frac{2(n+1)-1}{4n+2} \pi \right\}$$

$$= \prod_{j=1}^{n} \left\{ 1 - 2i \cos \frac{(2j-1)\pi}{2(2n+1)} \right\} \cdot \prod_{k=n+2}^{2n+1} \left\{ 1 - 2i \cos \frac{(2k-1)\pi}{2(2n+1)} \right\}$$

Letting j = (2n + 2 - k) in the second product, we get

$$L_{2n+1} = \prod_{j=1}^{n}\left[1 - 2i\cos\frac{(2j-1)\pi}{2(2n+1)}\right]\cdot\prod_{j=1}^{n}\left[1 - 2i\cos\left\{\pi - \frac{(2j-1)\pi}{2(2n+1)}\right\}\right]$$

(1)

$$= \prod_{j=1}^{n}\left\{1 + 4\cos^2\frac{2j-1}{2n+1}\cdot\frac{\pi}{2}\right\}.$$

Similarly,

(2) $$L_{2n} = \prod_{j=1}^{n}\left\{1 + 4\cos^2\frac{2j-1}{2n}\cdot\frac{\pi}{2}\right\}.$$

Hence from (1) and (2) we have the required result.

Also solved by Charles Wall, Douglas Lind, and David Zeitlin.

VIVA LA DIFFÉRENCE

H-112 Proposed by L. Carlitz, Duke University, Durham, N. Carolina.

Show that, for $n \geq 1$,

a) $$L_{n+1}^5 - L_n^5 - L_{n-1}^5 = 5L_{n+1}L_nL_{n-1}(2L_n^2 - 5(-1)^n)$$

b) $$F_{n+1}^5 - F_n^5 - F_{n-1}^5 = 5F_{n+1}F_nF_{n-1}(2F_n^2 + (-1)^n)$$

c) $$L_{n+1}^7 - L_n^7 - L_{n-1}^7 = 7L_{n+1}L_nL_{n-1}(2L_n^2 - 5(-1)^n)^2$$

d) $$F_{n+1}^7 - F_n^7 - F_{n-1}^7 = 7F_{n+1}F_nF_{n-1}(2F_n^2 + (-1)^n)^2 .$$

Solution by the proposer.

For parts c) and d), take $x = L_n$, $y = L_{n-1}$ in the identity

$$(x+y)^7 - x^7 - y^7 = 7xy(x+y)(x^2 + xy + y^2)^2 .$$

Since

$$L_n^2 + L_nL_{n-1} + L_{n-1}^2 = 2L_n^2 - 5(-1)^n ,$$

we get

$$L_{n+1}^7 - L_n^7 - L_{n-1}^7 = 7L_{n+1}L_nL_{n-1}(2L_n^2 - 5(-1)^n)^2 .$$

Similarly, since

$$F_n^2 + F_nF_{n-1} + F_{n-1}^2 = 2F_n^2 + (-1)^n ,$$

we get

$$F_{n+1}^7 - F_n^7 - F_{n-1}^7 = 7F_{n+1}F_nF_{n-1}(2F_n^2 + (-1)^n)^2 .$$

Parts a) and b) follow in a similar manner, by selecting $x = L_n$, $y = L_{n-1}$; $x = F_n$, $y = F_{n-1}$ in the identity

$$(x + y)^5 - x^5 - y^5 = 5xy(x + y)(x^2 + xy + y^2) .$$

Also solved by Charles Wall.

MINOR EXPANSION

H-117 Proposed by George Ledin, Jr., San Francisco, Calif.

Prove

$$\begin{vmatrix} F_{n+3} & F_{n+2} & F_{n+1} & F_n \\ F_{n+2} & F_{n+3} & F_n & F_{n+1} \\ F_{n+1} & F_n & F_{n+3} & F_{n+2} \\ F_n & F_{n+1} & F_{n+2} & F_{n+3} \end{vmatrix} = F_{2n+6}\,F_{2n}$$

Solution by C. B. A. Peck, Ordnance Research Laboratory, State College, Pa.

The determinant (first evaluated in 1866)

$$\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix} = (a - b - c + d)\,(a - b + c - d)\,(a + b - c - d)(a+b+c+d) .$$

In this case the product is

$$F_n(F_{n+1} + F_{n-1})F_{n+3}(F_{n+4} + F_{n+2})$$

from the recurrence

$$F_{n+1} = F_n + F_{n-1} \, .$$

The identities

$$L_n = F_{n+1} + F_{n-1}$$

and

$$F_{2n} = F_n L_n$$

now complete the proof.

Also solved by David Zeitlin, A. Shannon, D. Jaiswal, J. Biggs, F. Parker, S. Lajos, H. Krishna, and Stanley Rabinowitz

GOOD COMBINATION

H-119 Proposed by L. Carlitz, Duke University, Durham, N. Carolina

Put

$$\overline{H}(m,n,p) = \sum_{i=0}^{m} \sum_{j=0}^{n} \sum_{k=0}^{p} (-1)^{i+j+k} \binom{i+j}{j} \binom{j+k}{k} \binom{k+m-i}{m-i} \binom{m-i+n-j}{n-j} \binom{n-j+p-k}{p-k} \binom{p-k+i}{i} \, .$$

Show that $\overline{H}(m,n,p) = 0$ unless m,n,p are all even, and that

$$\overline{H}(2m,2n,2p) = \sum_{r=0}^{\min(m,n,p)} (-1)^r \frac{(m+n+p-r)!}{r!\,r!\,(m-r)!\,(n-r)!\,(p-r)!} \, .$$

(The formula

$$\overline{H}(2m, 2n) = \binom{m+n}{m}^2 ,$$

where

$$\overline{H}(m, n) = \sum_{i=0}^{m} \sum_{j=0}^{n} (-1)^{i+j} \binom{i+j}{j}\binom{m-i+j}{j}\binom{i+n-j}{n-j}\binom{m-i+n-j}{n-j}$$

is proved in the Fibonacci Quarterly, Vol. 4 (1966), pp. 323-325.)

Solution by the proposer.

As a special case of a more general identity (SIAM Review, Vol. 6 (1964) pp. 20-30, formulas (3.1)), we have

$$\sum_{i_1,\cdots,i_6=0}^{\infty} \binom{i_1+i_2}{i_2}\binom{i_2+i_3}{i_3}\binom{i_3+i_4}{i_4}\binom{i_4+i_5}{i_5}\binom{i_5+i_6}{i_6}\binom{i_6+i_1}{i_1} u_1^{i_1} u_2^{i_2} u_3^{i_3} u_4^{i_4} u_5^{i_5} u_6^{i_6}$$

$$= \Big\{ \big[1 - u_1 - u_2 - u_3 - u_4 - u_5 - u_6 + u_1u_4 + u_1u_5 + u_2u_4 + u_2u_5 + u_2u_6 + u_3u_5 + u_3u_6 + u_4u_6 - u_1u_3u_5 - u_2u_4u_6\big]^2 - 4u_1u_2u_3u_4u_5u_6 \Big\}^{-\frac{1}{2}} .$$

In this identity, take

$$u_4 = -u_1,\ u_5 = -u_2,\ u_6 = -u_3 .$$

Changing the notation slightly we get

$$\sum_{m,n,p=o}^{\infty} \overline{H}(m,n,p)\, u^m v^n w^p = \left\{(1 - u^2 - v^2 - w^2)^2 + 4u^2 v^2 w^2\right\}^{-\frac{1}{2}}$$

$$= \sum_{r=0}^{\infty} (-1)^r \binom{2r}{r} \frac{u^{2r} v^{2r} w^{2r}}{(1 - u^2 - v^2 - w^2)^{2r+1}}$$

$$= \sum_{r=0}^{\infty} (-1)^r \binom{2r}{r} u^{2r} v^{2r} w^{2r} \sum_{n=0}^{\infty} \binom{2r+n}{n} (u^2+v^2+w^2)^n$$

$$= \sum_{r=0}^{\infty} (-1)^r \binom{2r}{r} u^{2r} v^{2r} w^{2r} \times$$

$$\times \sum_{i,j,k=0}^{\infty} \frac{(2r+i+j+k)!}{(2r)!\, i!\, j!\, k!} u^{2i} v^{2j} w^{2k}$$

$$= \sum_{m,n,p=0}^{\infty} u^{2m} v^{2n} w^{2p} \sum_{r=0}^{\min(m,n,p)} (-1)^r \times$$

$$\times \frac{(m+n+p-r)!}{r!\, r!\, (m-r)!\, (n-r)!\, (p-r)!}$$

Comparing coefficients we get

$$\overline{H}(2m, 2n, 2p) = \sum_{r=0}^{\min(m,n,p)} (-1)^r \frac{(m+n+p-r)!}{r!\, r!\, (m-r)!\, (n-r)!\, (p-r)!} \quad .$$

It does not seem possible to sum the series on the right.

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