GOLDEN TRIANGLES, RECTANGLES, AND CUBOIDS

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1. INTRODUCTION

One of the most famous of all geometric figures is the Golden Rectangle, which has the ratio of length to width equal to the Golden Section,

$$\phi = (1 + \sqrt{5})/2 \ .$$

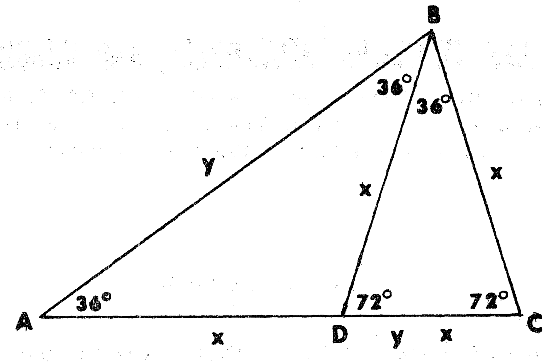
The proportions of the Golden Rectangle appear consistently throughout classical Greek art and architecture. As the German psychologists Fechner and Wundt have shown in a series of psychological experiments, most people do unconsciously favor "golden dimensions" when selecting pictures, cards, mirrors, wrapped parcels, and other rectangular objects. For some reason not fully known by either artists or psychologists, the Golden Rectangle holds great aesthetic appeal. Surprisingly enough, the best integral lengths to use for sides of an approximation to the Golden Rectangle are adjacent members of the Fibonacci series: 1, 1, 2, 3, 5, 8, 13,$\cdots$, and we find 3 x 5 and 5 x 8 filing cards, for instance.

Suppose that, instead of a Golden Rectangle, we study a golden section triangle. If the ratio of a side to the base is

$$\phi = (1 + \sqrt{5})/2 \ ,$$

then we will call the triangle a Golden Triangle. (See [2], [3].)

Now, consider the isosceles triangle with a vertex angle of 36°. On bisecting the base angle of 72°, two isosceles triangles are formed, and ΔBDC is similar to ΔABC as indicated in the figure:



Since $\Delta ABC \sim \Delta BDC$,

$$\frac{AB}{BD} = \frac{BC}{DC},$$

or,

$$\frac{y}{x} = \frac{x}{y - x},$$

so that

$$y^2 - yx - x^2 = 0.$$

Dividing through by $x^2 \neq 0$,

$$\frac{y^2}{x^2} - \frac{y}{x} - 1 = 0.$$

The quadratic equation gives

$$\frac{y}{x} = (1 + \sqrt{5})/2 = \phi$$

as the positive root, so that ΔABC is a Golden Triangle. Notice also, that, using the common altitude from B, the ratio of the area of ΔABC to ΔADB is ϕ.

Since the central angle of a regular decagon is 36°, ΔABC above shows that the ratio of the radius y to the side x of an inscribed decagon is

$$\phi = (1 + \sqrt{5})/2 .$$

Also, in a regular pentagon, the angle at a vertex between two adjacent diagonals is 36°. By reference to the figure above, the ratio of a diagonal to a side of a regular pentagon is also ϕ.

2. A TRIGONOMETRIC PROPERTY OF THE ISOSCELES GOLDEN TRIANGLE

The Golden Triangle with vertex angle 36° can be used for a surprising trigonometric application. Few of the trigonometric functions of an acute angle have values which can be expressed exactly. Usually, a method of approximation is used; most values in trigonometric tables cannot be expressed exactly as terminating decimals, repeating decimals, or even square roots, since they are approximations to transcendental numbers, which are numbers so irrational that they are not the root of any polynomial over the integers.

The smallest integral number of degrees for which the trigonometric functions of the angle can be expressed exactly is three degrees. Then, all multiples of 3° can also be expressed exactly by repeatedly using formulas such as $\sin(A + B)$. Strangely enough, the Golden Triangle can be used to derive the value of $\sin 3°$.

In our Golden Triangle, the ratio of the side to the base was

$$y/x = (1 + \sqrt{5})/2 .$$

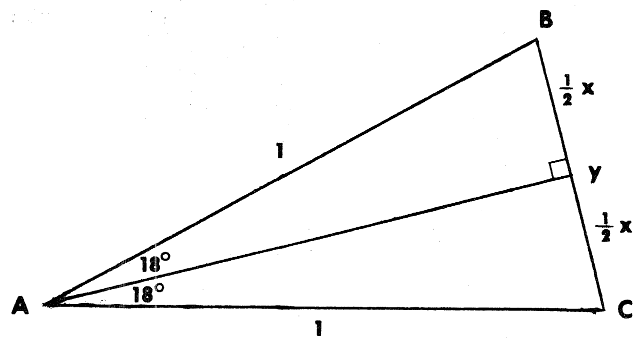
Suppose we let $AB = y = 1$. Then

$$1/x = (1 + \sqrt{5})/2 ,$$

or,

$$x = (\sqrt{5} - 1)/2 .$$

Redrawing the figure and bisecting the 36° angle,



we form right triangle AYC with YC = x/2. Then,

$$\sin 18^\circ = \frac{YC}{AC} = \frac{x}{2} = \frac{\sqrt{5}-1}{4} = \frac{1}{2\phi} .$$

Since $\sin^2 A + \cos^2 A = 1$,

$$\cos 18^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4} = \frac{\sqrt{\sqrt{5}\,\phi}}{2} .$$

Since $\sin(A - B) = \sin A \cos B - \sin B \cos A$,

$$\sin 15^\circ = \sin(45^\circ - 30^\circ) = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4} .$$

Similarly, using $\cos(A - B) = \cos A \cos B + \sin A \sin B$,

$$\cos 15^\circ = \frac{\sqrt{6}+\sqrt{2}}{4}$$

Again using the formula for $\sin(A - B)$,

$$\sin 3^\circ = \sin(18^\circ - 15^\circ) = \left(\frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right) - \left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right)$$
$$= \frac{1}{16}\left[(\sqrt{5}-1)(\sqrt{6}+\sqrt{2}) - 2(\sqrt{3}-1)\left(\sqrt{5+\sqrt{5}}\right)\right]$$

as given by Ransom in [1].

3. GOLDEN RECTANGLE AND GOLDEN TRIANGLE THEOREMS

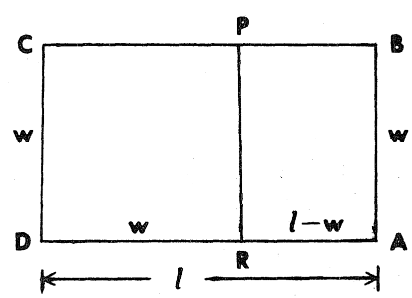
While a common way to describe the Golden Rectangle is to give the ratio of length to width as

$$\phi = (1 + \sqrt{5})/2 ,$$

this ratio is a consequence of the geometric properties of the Golden Rectangle which are discussed in this section.

Theorem. Given that the ratio of length to width of a rectangle is $k > 1$. A square with side equal to the width can be removed to leave a rectangle similar to the original rectangle if and only if $k = (1 + \sqrt{5})/2$.

Proof. Let the square PCDR be removed from rectangle ABCD, leaving rectangle BPRA.



If rectangles ABCD and BPRA have the same ratio of length to width, then

$$k = \frac{w}{l - w} = \frac{l}{w} .$$

Cross-multiplying and dividing by $w^2 \neq 0$ gives a quadratic equation in $\frac{l}{w}$ which has

$$(1 + \sqrt{5})/2$$

as its positive root. If

$$\frac{l}{w} = (1 + \sqrt{5})/2 = \phi,$$

then

$$\frac{w}{l - w} = \frac{1}{\frac{l}{w} - 1} = \frac{1}{\phi - 1} = \phi$$

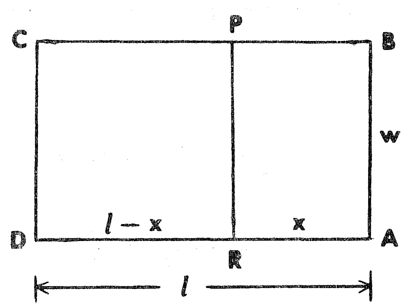
so that both rectangles have the same ratio of length to width.

Theorem. Given that the ratio of length to width of a rectangle is $k > 1$. A rectangle similar to the first can be removed to leave a rectangle such that the ratio of the areas of the original rectangle and the rectangle remaining is k, if and only if

$$k = (1 + \sqrt{5})/2 \quad .$$

Further, the rectangle remaining is a square.

Proof. Remove rectangle BPRA from rectangle ABCD as in the figure:



Then

$$\frac{\text{area ABCD}}{\text{area PCDR}} = \frac{lw}{w(l - x)} \quad .$$

But,

$$\frac{l w}{w(l - x)} = \frac{l}{w} = k \quad ,$$

if and only if

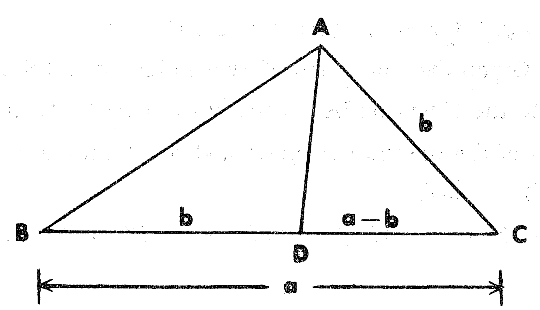
$$\frac{w}{l - x} = 1 \quad ,$$

or $w = l - x$ or PCDR is a square. Thus, our second theorem is a consequence of the first theorem.

Analogous theorems hold for Golden Triangles.

Theorem. Given that the ratio of two sides a and b of a triangle is $a/b = k > 1$. A triangle with side equal to b can be removed to leave a triangle similar to the first if and only if $k = (1 + \sqrt{5})/2$.

Proof. Remove ΔABD from ΔABC.



If $\Delta ADC \sim \Delta BAC$, then

$$\frac{AC}{BC} = \frac{DC}{AC}$$

or

$$\frac{b}{a} = \frac{a - b}{b} \quad .$$

Cross multiply, divide by $b^2 \neq 0$, and solve the quadratic in a/b to give

$$a/b = (1 + \sqrt{5})/2$$

as the only positive root.

If

$$a/b = (1 + \sqrt{5})/2 ,$$

then

$$DC/AC = (a - b)/b = a/b - 1 = (\sqrt{5} - 1)/2$$

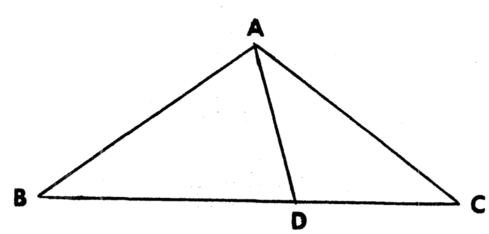
and

$$AC/BC = b/a = 2/(1 + \sqrt{5}) = (\sqrt{5} - 1)/2 = DC/AC .$$

Since $\angle C$ is in both triangles, $\Delta ADC \sim \Delta BAC$.

Theorem. Given that the ratio of two sides of a triangle is $k > 1$. A triangle similar to the first can be removed to leave a triangle such that the ratio of the areas of the original triangle and the triangle remaining is k, if and only if $k = (1 + \sqrt{5})/2$.

Proof. Let $\Delta ADC \sim \Delta BAC$, such that $BC/AC = AC/DC = k$.



If the ratio of areas of the original triangle and the one remaining is k, since there is a common altitude from A,

$$k = \frac{\text{area } \Delta BAC}{\text{area } \Delta BDA} = \frac{(BC)(h/2)}{(BC - DC)(h/2)} = \frac{BC/AC}{BC/AC - DC/AC} = \frac{k}{k - 1/k} .$$

Again cross-multiplying and solving the quadratic in k gives $k = (1 + \sqrt{5})/2$.

If

$$k = (1 + \sqrt{5})/2 ,$$

then

$$BC/AC = AC/DC = (1 + \sqrt{5})/2 ,$$

and the ratio of areas $BC/(BC - DC)$ becomes $(1 + \sqrt{5})/2$ when divided through by AC and then simply substituting the values of BC/AC and DC/AC.

If

$$k = (1 + \sqrt{5})/2 = BC/AC ,$$

and the ratio of areas of ΔBAC and ΔBDA is also k, then

$$k = \frac{BC/AC}{BC/AC - DC/AC} = \frac{k}{k - x} ,$$

which leads to

$$x = k - 1 \quad \text{or} \quad DC/AC = (1 + \sqrt{5})/2 - 1 = 2/(1 + \sqrt{5})$$

so that

$$AC/DC = (1 + \sqrt{5})/2$$

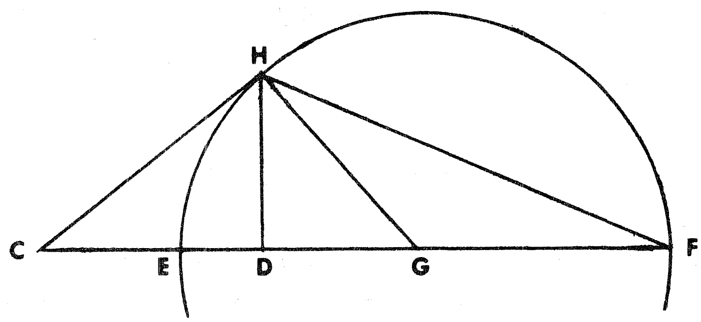
and ΔBAC is similar to ΔADC.

4. THE GENERAL GOLDEN TRIANGLE

Unlike the Golden Rectangle, the Golden Triangle does not have a unique shape. Consider a line segment $\overline{CD}$ of length

$$\phi = (1 + \sqrt{5})/2 .$$

Place points E, G, and F on line $\overleftrightarrow{CD}$ such that CE = 1, EG = GF = ϕ as in the diagram.



Then, ED = ϕ - 1 and

$$CE/ED = 1(\phi - 1) = \phi,$$

$$CF/DF = (2\phi + 1)(\phi + 1) = \phi^3/\phi^2 = \phi ,$$

so that E and F divide segment $\overline{CD}$ internally and externally in the ratio ϕ. Then the circle with center G is the circle of Apollonius for $\overline{CD}$ with ratio ϕ. Incidentally, the circle through C, D, and H is orthogonal to circle with center G and passing through H, and $\overline{HG}$ is tangent to the circle through C, D, and H.

Let H be any point on the circle of Apollonius. Then CH/HD = ϕ, CG/HG = ϕ, and ΔCHG ~ ΔHDG. The area of ΔCHG is

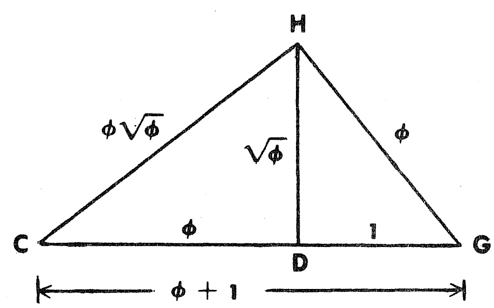
$$h(1 + \phi)/2 = h\phi^2/2 ,$$

and when ΔHDG is removed, the area of the remaining ΔCHD is $h\phi/2$, so that the areas have ratio ϕ. Then, ΔCHG is a Golden Triangle, and there are an infinite number of Golden Triangles because H can take an infinite number of positions on circle G.

If we choose H so that CH = $\phi + 1$, then we have the isosceles 36-72-72 Golden Triangle of decagon fame. If we erect a perpendicular at D and let H be the intersection with the circle of Apollonius, then we have a right golden triangle by applying the Pythagorean theorem and its converse. In our right golden triangle ΔCHG, CH = $\phi\sqrt{\phi}$, HG = ϕ, and CG = ϕ^2. The two smaller right triangles formed by the altitude to $\overline{CG}$ are each similar to ΔCHG, so that all three triangles are golden. The areas of ΔHDG, ΔCDH, and ΔCHG form the geometric progression,

$$\sqrt{\phi}/2, \quad (\sqrt{\phi}/2)\phi, \quad (\sqrt{\phi}/2)\phi^2 \ .$$

Before going on, notice that the right golden triangle ΔCHG provides an unusual and surprising configuration. While two pairs of sides and all three pairs of angles of ΔCHG and ΔCDH are congruent, yet ΔCHG is not congruent to ΔCDH! Similarly for ΔCDH and ΔHDG. (See Holt [4].)



5. THE GOLDEN CUBOID

H. E. Huntley [5] has described a Golden Cuboid (rectangular parallelopiped) with lengths of edges a, b, and c, such that

$$a : b : c = \phi : 1 : \phi^{-1} \ .$$

The ratios of the areas of the faces are

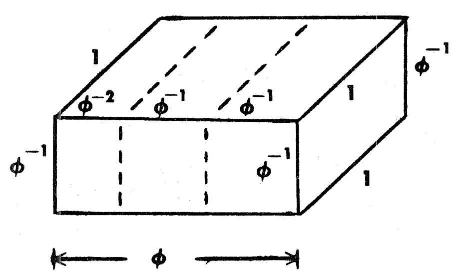
$$\phi : 1 : \phi^{-1} \ ,$$

and four of the six faces of the cuboid are Golden Rectangles.

If two cuboids of dimension

$$\phi^{-1} \times 1 \times \phi^{-1}$$

are removed from the Golden Cuboid, the remaining cuboid is similar to the original and is also a golden cuboid,



If a cuboid similar to the original is removed and has sides b, c, and d, then

$$b : c : d = \phi$$

so that

$$c = d\phi, \quad b = d\phi^2, \quad a = d\phi^3 \ .$$

The volume of the original is $abc = \phi^6 d^3$, and the volume removed is $bcd = \phi^3 d^3$. The remaining volume is $(\phi^6 - \phi^3)d^3$. The ratio of the volume of the original to the volume of the remaining cuboid is

$$\frac{\phi^6 d^3}{(\phi^6 - \phi^3)d^3} = \frac{\phi^3}{\phi^3 - 1} = \frac{2 + \sqrt{5}}{1 + \sqrt{5}} = \frac{3 + \sqrt{5}}{4} = \phi^2/2$$

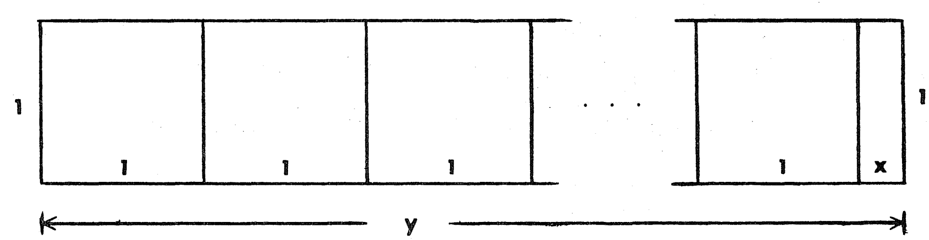
6. LUCAS GOLDEN-TYPE RECTANGLES

Now, in a Golden Rectangle, if one square with side equal to the width is removed, the resulting rectangle is similar to the original. Suppose that we have a rectangle in which when k squares equal to the width are removed, a rectangle similar to the original is formed, as discussed by J. A. Raab [6]. In the figure below, the ratio of length to width in the original rectangle and in the similar one formed after removing k squares is $y:1 = 1:x$ which gives $x = 1/y$. Since each square has side 1,

$$y - x = y - 1/y = k,$$

or,

$$y^2 - ky - 1 = 0 .$$



Let us consider only Lucas golden-type rectangles. That is, let $k = L_{2m+1}$, where L_{2m+1} is the $(2m+1)^{st}$ Lucas number defined by

$$L_0 = 2,\ L_1 = 1,\ L_n = L_{n-1} + L_{n-2} .$$

A known identity is

$$L_k = \left(\frac{1+\sqrt{5}}{2}\right)^k + \left(\frac{1-\sqrt{5}}{2}\right)^k = \alpha^k + \beta^k ,$$

where α and β are the roots of $x^2 - x - 1 = 0$.

In our problem, if

$$k = L_{2m+1} ,$$

then

$$y^2 - ky - 1 = 0$$

becomes

$$y^2 - L_{2m+1} y - 1 = 0$$

so that

$$y = \alpha^{2m+1}$$

or

$$y = \beta^{2m+1} ,$$

but

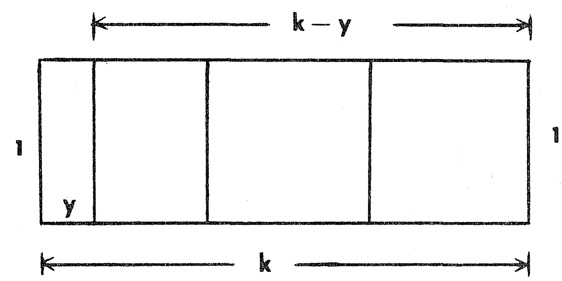
$$y = \alpha^{2m+1}$$

is the only positive root. Then

$$x = 1/\alpha^{2m+1} = -\beta^{2m+1} .$$

On the other hand, suppose we insist that to a given rectangle we add one similar to it such that the result is k squares long. Illustrated for $k = 3$, the equal ratios of length to width in the similar rectangles gives

$$\frac{1}{y} = \frac{k - y}{1} \quad \text{or} \quad ky - y^2 = 1 \quad \text{or} \quad y^2 - ky + 1 = 0.$$



Now, let $k = L_{2m}$; then $y = \alpha^{2m}$ or $y = \beta^{2m}$. Here, of course, $y = \beta^{2m}$, so that

$$k - y = L_{2m} - \beta^{2m} = \alpha^{2m} .$$

Both of these cases are, of course, in the plane; the reader is invited to extend these ideas into the third dimension.

7. GENERALIZED GOLDEN-TYPE CUBOIDS

Let the dimensions of a cuboid be $a : b : c = k$ and remove a cuboid similar to the first with dimensions $b : c : d = k$. Then

$$c = dk, \qquad b = dk^2, \qquad a = dk^3 .$$

The volume of the original is

$$abc = k^6 d^3 ,$$

the volume removed is

$$bcd = k^3 d^3 ,$$

and the remaining volume is

$$(k^6 - k^3) d^3 .$$

The ratio of the original volume to that remaining is

$$\frac{k^6 d^3}{(k^6 - k^3)d^3} = \frac{k^3}{k^3 - 1} .$$

Now, let this ratio equal

$$k^2/L_0 = k^2/2 ,$$

which leads to

$$0 = k^3 - 2k - 1 = (k + 1)(k^2 - k - 1)$$

with roots

$$k = -1, \quad (1 \pm \sqrt{5})/2 ,$$

and having

$$k = (1 + \sqrt{5})/2$$

as its only positive root.

Now consider a hypercuboid in a hyperspace of 6 dimensions, with dimensions $a : b : c : d : e : f = k$. Remove a hypercuboid of dimensions

$$b : c : d : e : f : g = k ,$$

and the ratio of the original volume to the volume remaining is

$$\frac{abcdef}{abcdef - bcdefg} = \frac{g^6 k^{21}}{g^6 (k^{21} - k^{15})} = \frac{k^6}{k^6 - 1} ,$$

since

$$f = kg, \quad e = k^2 g, \quad d = k^3 g, \quad e = k^4 g, \quad b = k^5 g, \quad a = k^6 g .$$

Now set this ratio equal to k^2/L_3 or,

$$\frac{k^6}{k^6 - 1} = \frac{k^3}{4}$$

which leads to

$$k^6 - 4k^3 - 1 = 0$$

with roots

$$k = \alpha, \omega\alpha, \omega^2\alpha, \beta, \omega\beta, \omega^2\beta,$$

where ω and ω^2 are cube roots of unity. Then

$$k = \alpha = (1 + \sqrt{5})/2$$

is the only positive real root.

Suppose we have a cuboid in a hyperspace of $4m + 2$ dimensions. Let this have edges

$$a_1, a_2, a_3, \cdots, a_{4m+2},$$

and cut off a cuboid similar to it so that

$$k = a_1 = a_2 : a_3 : \cdots : a_{4m+2} = a_2 : a_3 : a_4 : \cdots : a_{4m+2} : a_{4m+3}$$

This implies that the dimensions are related by

$$a_n = k^{4m+3-n} a_{4m+3}$$

for $n = 1, 2, \cdots, 4m + 3$. The volume of the original cuboid is now $a_1 a_2 a_3 \cdots a_{4m+2}$ while the volume of the cuboid cut off is $a_2 a_3 \cdots a_{4m+2} a_{4m+3}$. The remaining cuboid has volume equal to the difference of these, making the ratio of the original volume to that remaining

$$\frac{a_1 a_2 a_3 \cdots a_{4m+2}}{a_2 a_3 \cdots a_{4m+2}(a_1 - a_{4m+3})} = \frac{a_1}{a_1 - a_{4m+3}} = \frac{k^{4m+2}}{k^{4m+2} - 1}$$

Now let us let this volume ratio equal to

$$k^{2m+1} / L_{2m+1} ,$$

where L_{2m+1} is the $(2m+1)^{st}$ Lucas number, yielding

$$k^{4m+2} - L_{2m+1} k^{2m+1} - 1 = 0$$

whose only positive root is

$$\alpha = (1 + \sqrt{5})/2 .$$

The proof is very neat. Since $\alpha\beta = -1$ for α and β the roots of $x^2 - x - 1 = 0$ and since $L_n = \alpha^n + \beta^n$, we can write

$$-1 = \alpha^{2m+1}(\alpha^{2m+1} + \beta^{2m+1}) - \alpha^{4m+2} = \alpha^{2m+1} L_{2m+1} - \alpha^{4m+2} ,$$

and rearrange the terms above to give

$$\begin{aligned} k^{4m+2} - L_{2m+1} k^{2m+1} - 1 &= (k^{4m+2} - \alpha^{4m+2}) - L_{2m+1}(k^{2m+1} - \alpha^{2m+1}) \\ &= (k^{2m+1} - \alpha^{2m+1})(k^{2m+1} + \alpha^{2m+1} - L_{2m+1}) \\ &= (k^{2m+1} - \alpha^{2m+1})(k^{2m+1} - \beta^{2m+1}) = 0 . \end{aligned}$$

Thus, $k = \alpha\omega_j, \beta\omega_j$, where ω_j are the $(2m+1)^{st}$ roots of unity, so that

$$k = \alpha = (1 + \sqrt{5})/2$$

is the only positive real root.

Now, let us return to the volumes of the cuboids in the hyperspace of $4m + 2$ dimensions. Let us set $a = a_{4m+3}$. Then, since $k = \alpha$, the volume of the original cuboid is

$$V_1 = a_1a_2 \cdots a_{4m+2} = a^{4m+2}\alpha^1\alpha^2\alpha^3 \cdots \alpha^{4m+2}$$

and the volume of the cuboid removed is

$$V_2 = a_2a_3 \cdots a_{4m+2}a_{4m+3} = a^{4m+2}\alpha^1\alpha^2\alpha^3 \cdots \alpha^{4m+1}$$

making the volume of the cuboid remaining

$$V_1 - V_2 = a^{4m+2}\alpha^{T_{4m+1}}(\alpha^{4m+2} - 1)$$

where T_n is the n^{th} triangular number. But,

$$\alpha^{4m+2} - 1 = L_{2m+1}\alpha^{2m+1}$$

so that the remaining cuboid is made up of L_{2m+1} square cuboids with total volume

$$a^{4m+2}\alpha^1\alpha^2\alpha^3 \cdots \alpha^{4m+1}(L_{2m+1}\alpha^{2m+1}) .$$

Thus we have generalized the Golden Cuboid of Huntley [5] and also the golden-type rectangle of Raab [6].

REFERENCES

1. William R. Ransom, Trigonometric Novelties, J. Weston Walch, Portland, Maine, 1959, pp. 22-23.
2. Verner E. Hoggatt, Jr., Fibonacci and Lucas Numbers, Houghton Mifflin Company, Boston, 1969.

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