PRODUCTS OF FIBONACCI AND LUCAS NUMBERS

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Let U_{x_i} denote a Fibonacci or a Lucas number and consider the product

$$U_{x_1} U_{x_2} \cdots U_{x_n} .$$

We are interested in finding a general method by which this product may be "expanded," i.e., expressed as a linear function of Fibonacci or Lucas numbers.

Beginning with the case in which $n = 2$ we find that there are four types of such products. Using Binet's formulas it is easily verified that these may be expressed as follows:

$$F_{x_1} L_{x_2} = F_{x_1+x_2} + (-1)^{x_2} F_{x_1-x_2}$$

$$L_{x_1} F_{x_2} = F_{x_1+x_2} - (-1)^{x_2} F_{x_1-x_2}$$

$$L_{x_1} L_{x_2} = L_{x_1+x_2} + (-1)^{x_2} L_{x_1-x_2}$$

$$F_{x_1} F_{x_2} = \frac{1}{5}\left[L_{x_1+x_2} - (-1)^{x_2} L_{x_1-x_2}\right].$$

From these four identities we make the following observations.

This "multiplication" is not commutative.

The product of a mixed pair (i.e., one factor is a Fibonacci number and the other is a Lucas number) is a linear function of Fibonacci numbers. The product of a Fibonacci and Lucas number is a function of Lucas numbers.

The coefficient of the second term is $(-1)^{x_2}$ or $-(-1)^{x_2}$ according as x_2 comes from the subscript of a Lucas or a Fibonacci number.

The factor $1/5$ occurs when both numbers in the product are Fibonacci.

For convenience we denote -1 by ϵ. Now consider ϵ^{x_i} as playing a dual role. As a coefficient of L_x or F_x it has the value $(-1)^{x_i}$. As an operator applied to these numbers it reduces their subscripts by $2x_i$. With this in mind, we may write

$$F_{x_1} L_{x_2} = (1 + \epsilon^{x_2}) F_{x_1+x_2} = F_{x_1+x_2} + (-1)^{x_2} F_{x_1-x_2}$$

$$L_{x_1} F_{x_2} = (1 - \epsilon^{x_2}) F_{x_1+x_2} = F_{x_1+x_2} - (-1)^{x_2} F_{x_1-x_2}$$

$$L_{x_1} L_{x_2} = (1 + \epsilon^{x_2}) L_{x_1+x_2} = L_{x_1+x_2} + (-1)^{x_2} L_{x_1-x_2}$$

$$F_{x_1} F_{x_2} = (1 - \epsilon^{x_2}) L_{x_1+x_2} = \frac{1}{5}\left[L_{x_1+x_2} - (-1)^{x_2} L_{x_1-x_2}\right].$$

We turn now to products containing three factors such as $L_{x_1} L_{x_2} F_{x_3}$.

For the moment we shall understand that $L_{x_1} L_{x_2} F_{x_3}$ means $(L_{x_1} L_{x_2}) F_{x_3}$. Then, making use of the above results, we have

$$\begin{aligned}
(L_{x_1} L_{x_2}) F_{x_3} &= \left[L_{x_1+x_2} + (-1)^{x_2} L_{x_1-x_2}\right] F_{x_3} \\
&= L_{x_1+x_2} F_{x_3} + (-1)^{x_2} L_{x_1-x_2} F_{x_3} \\
&= F_{x_1+x_2+x_3} - (-1)^{x_3} F_{x_1+x_2-x_3} + (-1)^{x_2} \times \\
&\qquad \times \left[F_{x_1-x_2+x_3} - (-1)^{x_3} L_{x_1-x_2-x_3}\right] \\
&= F_{x_1+x_2+x_3} - (-1)^{x_3} F_{x_1+x_2-x_3} + (-1)^{x_2} \times \\
&\qquad \times F_{x_1-x_2+x_3} - (-1)^{x_2+x_3} L_{x_1-x_2-x_3} .
\end{aligned}$$

Using ϵ^{x_i} we arrive at the same result.

$$
\begin{aligned}
L_{x_1}L_{x_2}F_{x_3} &= (1+\epsilon^{x_2})(1-\epsilon^{x_3})F_{x_1+x_2+x_3} \\
&= (1+\epsilon^{x_2})F_{x_1+x_2+x_3} - (-1)^{x_3}F_{x_1+x_2-x_3} \\
&= F_{x_1+x_2+x_3} - (-1)^{x_3}F_{x_1+x_2-x_3} + (-1)^{x_2}F_{x_1-x_2+x_3} - (-1)^{x_2+x_3} \times \\
&\qquad \times F_{x_1-x_2-x_3} .
\end{aligned}
$$

Since

$$
(1+\epsilon^{x_2})(1-\epsilon^{x_3}) = 1 + \epsilon^{x_2} - \epsilon^{x_3} - \epsilon^{x_2+x_3} ,
$$

we could proceed as follows:

$$
\begin{aligned}
L_{x_1}L_{x_2}F_{x_3} &= (1+\epsilon^{x_2} - \epsilon^{x_3} - \epsilon^{x_2+x_3})F_{x_1+x_2+x_3} \\
&= F_{x_1+x_2+x_3} + (-1)^{x_2}F_{x_1-x_2+x_3} - (-1)^{x_3}F_{x_1+x_2-x_3} - (-1)^{x_2+x_3} \times \\
&\qquad \times F_{x_1-x_2-x_3} .
\end{aligned}
$$

We leave it as an exercise to show that $L_{x_1}(L_{x_2}F_{x_3})$ when expanded by any of these methods leads to the same result.

There are eight types of products, each consisting of three factors. We list them below.

$$
F_{x_1}L_{x_2}L_{x_3} = (1+\epsilon^{x_2})(1+\epsilon^{x_3})F_{x_1+x_2+x_3}
$$

$$
L_{x_1}F_{x_2}L_{x_3} = (1-\epsilon^{x_2})(1+\epsilon^{x_3})F_{x_1+x_2+x_3}
$$

$$
L_{x_1}L_{x_2}F_{x_3} = (1+\epsilon^{x_2})(1-\epsilon^{x_3})F_{x_1+x_2+x_3}
$$

$$F_{x_1}F_{x_2}F_{x_3} = \frac{1}{5}(1-\epsilon^{x_2})(1-\epsilon^{x_3})F_{x_1+x_2+x_3}$$

$$L_{x_1}F_{x_2}F_{x_3} = \frac{1}{5}(1-\epsilon^{x_2})(1-\epsilon^{x_3})L_{x_1+x_2+x_3}$$

$$F_{x_1}L_{x_2}F_{x_3} = \frac{1}{5}(1+\epsilon^{x_2})(1-\epsilon^{x_3})L_{x_1+x_2+x_3}$$

$$F_{x_1}F_{x_2}L_{x_3} = \frac{1}{5}(1-\epsilon^{x_2})(1+\epsilon^{x_3})L_{x_1+x_2+x_3}$$

$$L_{x_1}L_{x_2}L_{x_3} = (1+\epsilon^{x_2})(1+\epsilon^{x_3})L_{x_1+x_2+x_3} \quad .$$

The preceding results are the bases for the following conjecture.

Let U_{x_i} represent a Fibonacci or a Lucas number. Let p be the number of Fibonacci numbers in a product of both Fibonacci and Lucas numbers. Let

$$\overline{U}_{x_1+x_2+\cdots+x_n}$$

denote a Fibonacci or a Lucas number according as p is odd or even. As a coefficient ϵ^{x_i} has the numerical value $(-1)^{x_i}$ but as an operator applied to

$$\overline{U}_{x_1+x_2+\cdots+x_n} \ ,$$

it reduces the subscript of the latter by $2x_i$.

Use

$$(1-\epsilon^{x_i}) \qquad \text{or} \qquad (1+\epsilon^{x_i})$$

according as x_i is the subscript of a Fibonacci or a Lucas number in the product. Then

$$\prod_{i=1}^{u} U_{x_i} = \frac{1}{5^{\left[\frac{p}{2}\right]}} (1 \pm \epsilon^{x_2})(1 \pm \epsilon^{x_3}) \cdots (1 \pm \epsilon^{x_n})\overline{U}_{x_1+x_2+\cdots+x_n} .$$

The proof of this conjecture is given at the end of this article. The following example will illustrate

$$\begin{aligned} F_{15}F_{12}L_{10}F_8 &= \frac{1}{5}(1 - \epsilon^{12})(1 + \epsilon^{10})(1 - \epsilon^{8})F_{45} \\ &= \frac{1}{5}(1 - \epsilon^{12})(1 + \epsilon^{10})(F_{45} - F_{29}) \\ &= \frac{1}{5}(1 - \epsilon^{12})(F_{45} - F_{29} + F_{25} - F_9) \\ &= \frac{1}{5}(F_{45} - F_{29} + F_{25} - F_9 - F_{21} + F_5 - F_1 + F_{-15}) \\ &= \frac{1}{5}(F_{45} - F_{29} + F_{25} - F_{21} + F_{15} - F_9 + F_5 - F_1) . \end{aligned}$$

The above rule also applies if the product consists entirely of Fibonacci or of Lucas numbers each with the same subscript. For example,

$$\begin{aligned} L_x^5 &= (1 + \epsilon^x)^4 L_{5x} \\ &= (1 + 4\epsilon^x + 6\epsilon^{2x} + 4\epsilon^{3x} + \epsilon^{4x})L_{5x} \\ &= L_{5x} + 4(-1)^x L_{3x} + 6(-1)^{2x}L_x + 4(-1)^{3x}L_{-x} + (-1)^{4x}L_{-3x} \\ &= L_{5x} + \left[4(-1)^x + (-1)^x\right] L_{3x} + \left[6(-1)^{2x} + 4(-1)^{2x}\right] L_x \\ &= L_{5x} + 5(-1)^x L_{3x} + 10\, L_x . \end{aligned}$$

More generally, if n is an odd integer we have

$$L_x^n = (1 + \epsilon^x)^{n-1} L_{nx}$$

$$= L_{nx} + \binom{n-1}{1}\epsilon^x L_{(n-2)x} + \binom{n-1}{2}\epsilon^{2x} L_{(n-4)x} + \cdots$$

$$+ \binom{n-1}{n-2}\epsilon^{(n-2)x} L_{-(n-4)x} + \binom{n-1}{n-1}\epsilon^{(n-1)x} L_{-(n-2)x}$$

Since

$$L_{-k} = (-1)^k L_k ,$$

we get

$$L_x^n = L_{nx} + \left[\binom{n-1}{1} + \binom{n-1}{n-1}\right]\epsilon^x L_{(n-2)x} + \left[\binom{n-1}{2} + \binom{n-1}{n-2}\right]\epsilon^{2x} L_{(n-4)x}$$

$$+ \cdots + \left[\binom{n-1}{\frac{n-1}{2}} + \binom{n-1}{\frac{n+1}{2}}\right]\epsilon^{\left(\frac{n-1}{2}\right)} L_x .$$

Making use of the identity

$$\binom{n}{m} + \binom{n}{n-m} = \binom{n+1}{m} ,$$

the last equation may be written

$$L_x^n = L_{nx} + \binom{n}{1}\epsilon^x L_{(n-2)x} + \binom{n}{2}\epsilon^{2x} L_{(n-4)x} + \cdots + \binom{n}{\frac{n-1}{2}}\epsilon^{\left(\frac{n-1}{2}\right)x} L_x$$

$$L_x^n = \sum_{i=0}^{\frac{n-1}{2}} (-1)^{xi}\binom{n}{i} L_{(n-2i)x} \qquad n = 1, 3, 5, \cdots .$$

Similarly, we get the following:

$$L_x^n = \sum_{i=0}^{\frac{n}{2}-1} \left[(-1)^{xi} \binom{n}{i} L_{(n-2i)x}\right] + 2(-1)^{\frac{n}{2}x} \binom{n-1}{\frac{n}{2}} \qquad (n,\ \text{even})$$

$$F_x^n = \frac{1}{5^{\frac{n-1}{2}}} \sum_{i=0}^{\frac{n-1}{2}} (-1)^{(x+1)i} \binom{n}{i} F_{(n-2i)x} \qquad (n,\ \text{odd})$$

$$F_x^n = \frac{1}{5^{\frac{n}{2}}} \sum_{i=0}^{\frac{n}{2}-1} \left[(-1)^{(x+1)i} \binom{n}{i} L_{(n-2i)x}\right] + 2(-1)^{\frac{n}{2}(x+1)} \binom{n-1}{\frac{n}{2}} \qquad (n,\ \text{even})$$

The proof of the rule which has been used to express products of Fibonacci and Lucas numbers as linear functions of those numbers is a proof by induction.

We have seen that it is true for $n = 2$ and $n = 3$. Assume it is true for all integral values of n up to and including k. Then, if p is even

$$\text{(1)} \qquad \prod_{i=1}^{k} U_{x_i} = \frac{1}{5^{\left[\frac{p}{2}\right]}} (1 \pm \epsilon^{x_2}) \cdots (1 \pm \epsilon^{x_k}) L_{x_1+x_2+\cdots+x_k} .$$

Multiplying both members of this equation by L_{x+1} we get

$$\prod_{i=1}^{k} U_{x_i} L_{x+1} = \frac{1}{5^{\left[\frac{p}{2}\right]}} (1 \pm \epsilon^{x_2}) \cdots (1 \pm \epsilon^{x_k}) L_{x_1+x_2+\cdots+x_k} L_{x_{k+1}}$$

$$= \frac{1}{5^{\left[\frac{p}{2}\right]}} (1 \pm \epsilon^{x_2}) \cdots (1 \pm \epsilon^{x_k}) \times$$

$$\times (L_{x_1+x_2+\cdots+x_{k+1}} + (-1)^{k+1} L_{x_1+x_2+\cdots+x_k-x_{k+1}})$$

$$= \frac{1}{5^{\left[\frac{p}{2}\right]}} (1 + \epsilon^{x_2}) \cdots (1 \pm \epsilon^{x_k})(1 + \epsilon^{x_{k+1}}) L_{x_1+x_2+\cdots+x_{k+1}}$$

Next, multiplying both sides of equation (1) by F_{x+1} we get

$$\prod_{i=1}^{k} U_{x_i} F_{x_{k+1}} = \frac{1}{5^{\left[\frac{p}{2}\right]}} (1 + \epsilon^{x_2}) \cdots (1 \pm \epsilon^{x_k}) L_{x_1+x_2+\cdots+x_k} F_{x_{k+1}}$$

$$= \frac{1}{5^{\left[\frac{p}{2}\right]}} (1 \pm \epsilon^{x_2}) \cdots (1 \pm \epsilon^{x_k}) \times$$

$$\times \left[F_{x_1+x_2+\cdots+x_{k+1}} - (-1)^{x_{k+1}} F_{x_1+x_2+\cdots+x_k-x_{k+1}} \right]$$

$$= \frac{1}{5^{\left[\frac{p}{2}\right]}} (1 \pm \epsilon^{x_2}) \cdots (1 \pm \epsilon^{x_k})(1 - \epsilon^{x_{k+1}}) F_{x_1+x_2+\cdots+x_{k+1}} .$$

Since both of these results agree with that given by the general rule for $n = k + 1$ the induction is complete for the case in which

$$\overline{U}_{x_1+x_2+\cdots+x_n} = L_{x_1+x_2+\cdots+x_n} .$$

We leave the case in which

$$\overline{U}_{x_1+x_2+\cdots+x_n} = F_{x_1+x_2+\cdots+x_n}$$

for the reader to prove.

We now consider the reverse problem; that is, the problem of finding a general method of expressing

$$L_{x_1+x_2+\cdots+x_n} \quad \text{and} \quad F_{x_1+x_2+\cdots+x_n}$$

as a homogeneous function of products, each of the type,

$$F_{x_1} F_{x_2} \cdots F_{x_i} L_{x_{i+1}} L_{x_{i+2}} \cdots L_{x_n} .$$

For simplicity let S_i^n denote the sum of all products consisting of i factors which are Fibonacci numbers and n - i which are Lucas numbers. The number of such factors is, of course, $\binom{n}{i}$.

For example,

$$S_2^4 = F_{x_1} F_{x_2} L_{x_3} L_{x_4} + F_{x_1} F_{x_3} L_{x_2} L_{x_4} + F_{x_1} F_{x_4} L_{x_2} L_{x_3} + F_{x_2} F_{x_3} L_{x_1} L_{x_4} + F_{x_2} F_{x_4} L_{x_1} L_{x_3} + F_{x_3} F_{x_4} L_{x_1} L_{x_2} \ .$$

For later use we note that

$$S_i^n L_{x_{n+1}} + S_{i-1}^n F_{x_{n+1}} = S_i^{n+1} \ .$$

This follows from the identity

$$\binom{n}{i} + \binom{n}{i-1} = \binom{n+1}{i} \ .$$

For the case n = 2 we readily prove (using Binet's formulas) that

$$F_{x_1+x_2} = \frac{1}{2} (L_{x_1} F_{x_2} + F_{x_1} L_{x_2})$$

$$= \frac{1}{2} S_1^2$$

$$L_{x_1+x_2} = \frac{1}{2} (L_{x_1} L_{x_2} + 5 F_{x_1} F_{x_2})$$

$$= \frac{1}{2} (S_0^2 + 5 S_2^2) \ .$$

Using these two identities as a basis, we develop the following for n = 3

$$
\begin{aligned}
F_{x_1+x_2+x_3} &= F_{(x_1+x_2)+x_3} \\
&= \frac{1}{2}\left[L_{x_1+x_2} F_{x_3} + F_{x_1+x_2} L_{x_3}\right] \\
&= \frac{1}{2}\left[\frac{1}{2}(L_{x_1} L_{x_2} + 5 F_{x_1} F_{x_2})F_{x_3} + \frac{1}{2}(L_{x_1} F_{x_2} + F_{x_1} L_{x_2}) L_{x_3}\right] \\
&= \frac{1}{2^2}\left[L_{x_1} L_{x_2} F_{x_3} + 5 F_{x_1} F_{x_2} F_{x_3} + L_{x_1} F_{x_2} L_{x_3} + F_{x_1} L_{x_2} L_{x_3}\right] \\
&= \frac{1}{2^2}\left[S_1^3 + 5 S_3^3\right]
\end{aligned}
$$

$$
\begin{aligned}
L_{x_1+x_2+x_3} &= L_{(x_1+x_2)+x_3} \\
&= \frac{1}{2}\left[L_{x_1+x_2} L_{x_3} + 5 F_{x_1+x_2} F_{x_3}\right] \\
&= \frac{1}{2}\left[\frac{1}{2}(L_{x_1} L_{x_2} + 5 F_{x_1} F_{x_2})L_{x_3} + \frac{5}{2}(L_{x_1} F_{x_2} + F_{x_1} L_{x_2}) F_{x_3}\right] \\
&= \frac{1}{2^2}\left[L_{x_1} L_{x_2} L_{x_3} + 5 F_{x_1} F_{x_2} L_{x_3} + 5 F_{x_1} L_{x_2} F_{x_3} + 5 L_{x_1} F_{x_2} F_{x_3}\right] \\
&= \frac{1}{2^2}\left[S_0^3 + 5 S_2^3\right] .
\end{aligned}
$$

Proceeding in this manner we derive the following identities for $n = 4$ and $n = 5$:

$$F_{x_1+x_2+x_3+x_4} = \frac{1}{2^3}\left[S_1^4 + 5 S_3^4\right]$$

$$F_{x_1+x_2+x_3+x_4+x_5} = \frac{1}{2^4}\left[S_1^5 + 5 S_3^5 + 5^2 S_5^5\right]$$

$$L_{x_1+x_2+x_3+x_4} = \frac{1}{2^3}\left[S_0^4 + 5 S_2^4 + 5^2 S_4^4\right]$$

$$L_{x_1+x_2+x_3+x_4+x_5} = \frac{1}{2^4}\left[S_0^5 + 5S_2^5 + 5^2 S_4^5\right].$$

From the above results we conjecture the validity of the following identities which we will prove later.

$$\text{(2)} \quad F_{x_1+x_2+\cdots+x_n} = \frac{1}{2^{n-1}}\left[S_1^n + 5S_3^n + 5^2 S_5^n + \cdots + \begin{cases} 5^{\frac{n-2}{2}} S_{n-1}^n \Big] & (n,\ \text{even}) \\ 5^{\frac{n-1}{2}} S_n^n \Big] & (n,\ \text{odd}) \end{cases}\right.$$

$$\text{(3)} \quad L_{x_1+x_2+\cdots+x_n} = \frac{1}{2^{n-1}}\left[S_0^n + 5S_2^n + 5^2 S_4^n + \cdots + \begin{cases} 5^{\frac{n}{2}} S_n^n \Big] & (n,\ \text{even}) \\ 5^{\frac{n-1}{2}} S_{n-1}^n \Big] & (n,\ \text{odd}) \end{cases}\right. .$$

Before proceeding with the proofs of these identities we consider the special case when $x_1 = x_2 = \cdots = x_n = x$. For this case we get the following:

$$F_{nx} = \frac{1}{2^{n-1}}\left[\binom{n}{1} F_x L_x^{n-1} + 5\binom{n}{3} F_x^3 L_x^{n-3} + \cdots + \begin{cases} 5^{\frac{n-2}{2}} \binom{n}{n-1} F_x^{n-1} L_x \Big] & (n,\text{even}) \\ 5^{\frac{n-1}{2}} \binom{n}{n} F_x^n \Big] & (n,\text{odd}) \end{cases}\right.$$

$$L_{nx} = \frac{1}{2^{n-1}}\left[L_x^n + 5\binom{n}{2} F_x^2 L_x^{n-2} + \cdots + \begin{cases} 5^{\frac{n}{2}} \binom{n}{n} F_x^n \Big] & (n,\ \text{even}) \\ 5^{\frac{n-1}{2}} \binom{n}{n-1} F_x^{n-1} L_x \Big] & (n,\ \text{odd}) \end{cases}\right.$$

Note, in particular, if $n = 2$ we get two well-known identities

$$F_{2x} = F_x L_x$$

and

$$L_{2x} = \frac{1}{2}(L_x^2 + 5F_x^2) .$$

We have now to prove the identities (1) and (2). The proof is by induction on n. Both identities are true for n = 2. We assume they are valid for all integral values of n up to and including n = k.

Then

$$(4) \quad F_{x_1+x_2+\cdots+x_k} = \frac{1}{2^{k-1}}\left[S_1^k + 5S_3^k + 5^2 S_5^k + \cdots + \begin{cases} 5^{\frac{k-2}{2}} S_{k-1}^k \Big] & (k,\ \text{even}) \\ 5^{\frac{k-1}{2}} S_k^k \Big] & (k,\ \text{odd}) \end{cases}\right.$$

$$(5) \quad L_{x_1+x_2+\cdots+x_k} = \frac{1}{2^{k-1}}\left[S_0^k + 5S_2^k + 5^2 S_4^k + \cdots + \begin{cases} 5^{\frac{k}{2}} S_k^k \Big] & (k,\ \text{even}) \\ 5^{\frac{k-1}{2}} S_{k-1}^k \Big] & (k,\ \text{odd}) \end{cases}\right.$$

Now

$$(6) \quad L_{x_1+x_2+\cdots+x_k+x_{k+1}} \equiv L_{(x_1+x_2+\cdots+x_k)+x_{k+1}}$$

$$= \frac{1}{2}\left[L_{x_1+x_2+\cdots+x_k} L_{x_{k+1}} + 5F_{x_1+x_2+\cdots+x_k} F_{x_{k+1}}\right].$$

Applying (4) and (5) to the right member of (6), we get

$$(7) \quad L_{x_1+x_2+\cdots+x_k} L_{x_{k+1}} = \frac{1}{2^{k-1}}\left[S_0^k L_{x_{k+1}} + 5S_2^k L_{x_{k+1}} + \cdots\right.$$

$$+ \begin{cases} 5^{\frac{k}{2}} S_k^k L_{x_{k+1}} \Big] & (k,\ \text{even}) \\ 5^{\frac{k-1}{2}} S_{k-1}^k L_{x_{k+1}} \Big] & (k,\ \text{odd}) \end{cases}$$

$$(8)\quad F_{x_1+x_2+\cdots+x_k} F_{x_{k+1}} = \frac{1}{2^{k-1}}\left[S_1^k F_{x_{k+1}} + 5 S_3^k F_{x_{k+1}} + \cdots + \begin{cases} 5^{\frac{k-2}{2}} S_{k-1}^k F_{x_{k+1}} \Big] & (k,\ \text{even}) \\ 5^{\frac{k-1}{2}} S_k^k F_{x_{k+1}} \Big] & (k,\ \text{odd}) \end{cases}\right. .$$

Substituting in (6) from (7) and (8) and regrouping we get the following:

$$\begin{aligned} L_{x_1+x_2+\cdots+x_{k+1}} &= S_0^{k+1} + 5\left(S_2^k L_{x_{k+1}} + S_1^k F_{x_{k+1}}\right) \\ &\quad + 5^2\left(S_4^k L_{x_{k+1}} + S_3^k F_{x_{k+1}}\right) + \cdots \\ &\quad + \begin{cases} 5^{\frac{k}{2}}\left(S_k^k L_{x_{k+1}} + S_{k-1}^k F_{x_{k+1}}\right) & (k,\ \text{even}) \\ 5^{\frac{k-1}{2}} S_k F_{k+1} & (k,\ \text{odd}) \end{cases} \end{aligned}$$

Hence

$$L_{x_1+x_2+\cdots+x_{k+1}} = S_0^{k+1} + 5 S_2^{k+1} + 5^2 S_4^{k+1} + \cdots + \begin{cases} 5^{\frac{k}{2}} S_k^{k+1} & (k+1,\ \text{even}) \\ 5^{\frac{k-1}{2}} S_{k+1}^{k+1} & (k+1,\ \text{odd}) \end{cases}$$

This completes the proof of (3). The proof of (2) is similar.

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ERRATA FOR PSEUDO-FIBONACCI NUMBERS

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Please make the following changes in the above-entitled article appearing in Vol. 6, No. 6:

p. 305: in Eq. (3), O_{i+1} should read: O_{i+2}. On p. 306, the 6th line from the bottom: B^{-k+1} should read: B^{k+1}. On page 310, in Eq. (12), $2O_{2n}$ should read: $2\lambda O_{2n}$; in Eq. (13), 30_{2n+1} should read: $3O_{2n+1}$. Equation (17), on p. 312: $(\lambda - 2)O_{2n-1}$ should read: $\lambda(\lambda - 2)O_{2n-1}$. Equation (18s) on p. 313: 40_i^2 should read: $4O_i^2$. In line 3, p. 314, 20_{2n+2} should read $2O_{2n+2}$, and Eq. (20), p. 315: $(\lambda - 2)O_{2n}$ should read $\lambda(\lambda - 2)O_{2n}$.

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