GENERATING FUNCTIONS

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1. INTRODUCTION

With an arbitrary sequence of (complex) numbers $\{a_n\} = \{a_0, a_1, a_2, \cdots\}$ we associate the (formal) power series

$$a(x) = \sum_{n=0}^{\infty} a_n x^n . \tag{1.1}$$

The definition is purely formal; convergence of the series need not be assumed. The series (1.1) is usually called an ordinary generating function.

Let $\{b_n\} = \{b_0, b_1, b_2, \cdots\}$ be another sequence and

$$b(x) = \sum_{n=0}^{\infty} b_n x^n$$

the corresponding generating function. We define the sum of $\{a_n\}$ and $\{b_n\}$ by means of

$$\{a_n\} + \{b_n\} = \{c_n\}, \quad c_n = a_n + b_n \quad (n = 0, 1, 2, \cdots) ;$$

then clearly

$$c(x) = \sum_{n=0}^{\infty} c_n x^n = a(x) + b(x) .$$

Similarly, if we define the product

$$\{a_n\}\{b_n\} = \{p_n\},$$

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by means of

$$p_n = \sum_{k=0}^{n} a_k b_{n-k} \quad (n = 0,1,2,\cdots) , \tag{1.2}$$

then it is easily seen that

$$p(x) = \sum_{n=0}^{\infty} p_n x^n = a(x)\, b(x) .$$

The product defined by (1.2) is called the Cauchy product of $\{a_n\}$ and $\{b_n\}$.

In contrast with (1.1) we may define the <u>exponential</u> generating function

$$A(x) = \sum_{n=0}^{\infty} a_n x^n/n! \tag{1.3}$$

which again is a formal definition. The product is now defined by means of

$$p_n = \sum_{k=0}^{\infty} \binom{n}{k} a_k b_{n-k} ; \tag{1.4}$$

this is known as the Hurwitz product and is of particular interest in certain number-theoretic questions (see for example [15, p. 147]).

One can develop an algebra of sequences using either the Cauchy or Hurwitz product. In either case multiplication is associative and commutative and distributive with respect to addition. Moreover the product of two sequences is equal to the zero sequence

$$\{z_n\} = \{0,0,0,\cdots\}$$

if and only if at least one factor is equal to $\{z_n\}$; thus the set of all sequences constitute a domain of integrity.

In the present paper, however, we shall be primarily interested in showing how generating functions can be employed to sum or transform finite series of various kinds. We shall also illustrate the use of generating functions in solving several enumerative problems. For a fuller treatment the reader is referred to [18].

In the definitions above we have considered only the case of one dimensional sequences. This can of course be generalized in an obvious way, namely with the double sequence $\{a_{m,n}\}$ we associate the series

$$a(x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n} x^m y^n . \tag{1.5}$$

Also factorials may be inserted as in (1.3). Indeed, there is now a certain amount of choice; for example both

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n} \frac{x^m y^n}{m!\,n!} , \quad \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n} x^m y^n/n! \tag{1.6}$$

are useful. As we shall see in Section 10, other possibilities also occur.

More generally, we may consider

$$a(x_1,\cdots,x_k) = \sum_{n_1,\cdots,n_k=0}^{\infty} a_{n_1,\cdots,n_k} x_1^{n_1} \cdots x_k^{n_k} \tag{1.7}$$

and its various modifications as in (1.6). Of particular interest in the theory of numbers is the Dirichlet series

$$\sum_{n=1}^{\infty} a_n /n^s ; \tag{1.8}$$

the product is now defined by

$$p_n = \sum_{rs=n} a_r b_s . \tag{1.9}$$

We may think of (1.8) as a generalization of (1.7). For let $q_1, q_2, \cdots, q_k$ denote the first k primes and let $a_n = 0$ unless

$$n = q_1^{f_1} q_2^{f_2} \cdots q_k^{f_k} .$$

If we put

$$a_n = a_{f_1, \cdots, f_k}$$

it follows that

$$\sum_{n=1}^{\infty} a_n / n^s = a(q_1^{-s}, \cdots, q_k^{-s}) , \tag{1.10}$$

where the right member is defined by (1.7).

2. As a first simple illustration of the generating function technique, we take the binomial expansion

$$(1 + x)^m = \sum_{k=0}^{m} \binom{m}{k} x^k , \tag{2.1}$$

where, to begin with, we assume m is a nonnegative integer. Combining (2.1) with

$$(1 + x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j$$

we immediately get

$$\sum_{s=0}^{k} \binom{m}{s} \binom{n}{k-s} = \binom{m+n}{k} \quad (k = 0, 1, 2, \cdots) . \tag{2.2}$$

It is to be understood that the binomial coefficient $\binom{n}{k} = 0$ if $k > n$ or $k < 0$.

Each side of (2.2) is a polynomial in m and n. Since (2.2) holds for all nonnegative values of m,n it follows that it holds when m,n are arbitrary complex numbers.

It is convenient to introduce the following notation:

$$(a)_n = a(a+1)\cdots(a+n-1), \quad (a)_0 = 1 .$$

It is easily verified that

$$\binom{a}{k} = (-1)^k \frac{(-a)_k}{k!} ,$$

and that (2.2) becomes

$$\sum_{s=0}^{\infty} \frac{(-k)_s (a)_s}{s!\,(b)_s} = \frac{(b-a)_k}{(b)_k} . \tag{2.3}$$

In (2.3) a and b are arbitrary except that b is not a negative integer.

The formula

$$\sum_{k=0}^{m} (-1)^{k-n} \binom{m}{n}\binom{k}{n} = \begin{cases} 1 & (m = n) \\ 0 & (m \neq n) \end{cases} \tag{2.4}$$

is very useful. The proof is quite simple. We may evidently assume $m \geq n$. Since

$$\binom{m}{k}\binom{k}{n} = \binom{m}{n}\binom{m-n}{k-n} ,$$

it is clear that the left member of (2.4) is equal to

$$\binom{m}{n} \sum_{k=n}^{m} (-1)^{k-n} \binom{m-n}{k-n} = \binom{m}{n} (-1)^{m-n}$$

and (2.4) follows at once.

As an immediate application of (2.4) we have the following theorem: If

$$b_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k \quad (n = 0,1,2,\cdots) \tag{2.5}$$

then

$$a_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} b_k \quad (n = 0,1,2,\cdots) \tag{2.6}$$

and conversely.

It is of interest to express the equivalence of (2.5) and (2.6) in terms of generating functions. As above, put

$$A(x) = \sum_{n=0}^{\infty} a_n x^n/n! , \quad B(x) = \sum_{n=0}^{\infty} b_n x^n/n! .$$

Then (2.5) becomes

$$B(x) = e^x A(-x) \tag{2.7}$$

while (2.6) becomes

$$A(x) = e^x B(-x) . \tag{2.8}$$

It is easy to extend the above to multiple sequences. If

$$a_{m,n} = \sum_{j=0}^{m} \sum_{k=0}^{n} (-1)^{j+k} \binom{m}{j} \binom{n}{k} b_{j,k} \tag{2.9}$$

then

$$b_{m,n} = \sum_{j=0}^{m} \sum_{k=0}^{n} (-1)^{j+k} \binom{m}{j} \binom{n}{k} a_{j,k} \tag{2.10}$$

and conversely. Moreover if

$$A(x,y) = \sum_{m,n=0}^{\infty} a_{m,n} \frac{x^m y^n}{m!\,n!}, \quad B(x,y) = \sum_{m,n=0}^{\infty} b_{m,n} \frac{x^m y^n}{m!\,n!}$$

then

$$A(x,y) = e^{x+y} B(-x,-y) \tag{2.11}$$

and

$$B(x,y) = e^{x+y} A(-x,-y) . \tag{2.12}$$

3. As a second illustration we shall prove the formula

$$\sum_{k=0}^{m} \binom{x}{k}\binom{n}{m-k}\binom{y+n-k}{n} = \sum_{k=0}^{m} \binom{y-x+n}{k}\binom{x}{m-k}\binom{y+n-k}{n-k}. \tag{3.1}$$

This result is a slight generalization of a formula due to Greenwood and Gleason [10] and Gould [9].

Put

$$A_{m,n} = \sum_{k=0}^{m} \binom{x}{k}\binom{n}{m-k}\binom{y+n-k}{n}, \quad B_{m,n} = \sum_{k=0}^{m} \binom{y-x+n}{k}\binom{x}{m-k}\binom{y+n-k}{n-k}.$$

Then

$$\begin{aligned}\sum_{m=0}^{\infty} A_{m,n} t^m &= \sum_{k=0}^{\infty} \binom{x}{k}\binom{y+n-k}{n} \sum_{m=k}^{n+k} \binom{n}{m-k} t^m \\ &= \sum_{k=0}^{\infty} \binom{x}{k}\binom{y+n-k}{n} t^k (1+t)^n ,\end{aligned}$$

$$\sum_{m,n=0}^{\infty} A_{m,n} t^m u^n = \sum_{k=0}^{\infty} \binom{x}{k} t^k \sum_{n=0}^{\infty} \binom{y+n-k}{n} (1+t)^n u^n$$

$$= \sum_{k=0}^{\infty} \binom{x}{k} t^k (1-u-tu)^{-y+k-1}$$

$$= (1-u-tu)^{-y-1} \sum_{k=0}^{\infty} \binom{x}{k} t^k (1-u-tu)^k$$

$$= (1-u-tu)^{-y-1} \left[1 + t(1-u-tu)\right]^x ,$$

so that

$$\sum_{m,n=0}^{\infty} A_{m,n} t^m u^n = \frac{(1+t)^x (1-tu)^x}{(1-u-tu)^{y+1}} . \tag{3.2}$$

On the other hand,

$$\sum_{m=0}^{\infty} B_{m,n} t^m = \sum_{k=0}^{n} \binom{y-x+n}{k} \binom{y+n-k}{n-k} \sum_{m=k}^{\infty} \binom{x}{m-k} t^m \tag{3.3}$$

$$= \sum_{k=0}^{n} \binom{y-x+n}{k} \binom{y+n-k}{n-k} t^k (1+t)^x ,$$

$$\frac{(1-tu)^x}{(1-u-tu)^{y+1}} = (1-tu)^{x-y-1} \left(1 - \frac{u}{1-tu}\right)^{-y-1}$$

$$= \sum_{r=0}^{\infty} \binom{y+r}{r} u^r (1-tu)^{x-y-r-1}$$

$$= \sum_{r=0}^{\infty} \binom{y+r}{r} u^r \sum_{k=0}^{\infty} \binom{y-x+r+k}{k} t^k u^k$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{y-x+n}{k} \binom{y+n-k}{n-k} t^k u^n ,$$

so that by (3.3),

$$\sum_{m,n=0}^{\infty} B_{m,n}\, t^m u^n = \frac{(1 + t)^x (1 - tu)^x}{(1 - u - tu)^{y+1}} \tag{3.4}$$

Comparing (3.4) with (3.2), (3.1) follows at once.

We remark that if we put

$$ {}_3F_2\left[\begin{matrix} a, b, c \\ d, e \end{matrix}\right] = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k (c)_k}{k!\,(d)_k (e)_k} $$

then (3.1) becomes

$$\binom{n}{m}\binom{y + n}{n}\, {}_3F_2\left[\begin{matrix} -x, -y, -m \\ -y-n, n-m+1 \end{matrix}\right] = \binom{x}{m}\binom{y + n}{n}\, {}_3F_2\left[\begin{matrix} x-y-n, -n, -m \\ -y-n, x-m+1 \end{matrix}\right],$$

which is a special case of a known transformation formula [1, p. 98, ex. 7].

4. A set of polynomials $A_n(x)$ that satisfy

$$A_n'(x) = n\, A_{n-1}(x) \quad (n = 0, 1, 2, \cdots), \tag{4.1}$$

where the prime denotes differentiation, is called an Appell set. It is easily proved that such a set may be defined by

$$\sum_{n=0}^{\infty} A_n(x)\, z^n/n! = e^{xz} \sum_{n=0}^{\infty} a_n\, z^n/n!, \tag{4.2}$$

where the a_n are independent of x. Also it is evident from (4.1) that

$$A_n(x) = \sum_{k=0}^{n} \binom{n}{k} a_k\, x^{n-k}. \tag{4.3}$$

This formula is sometimes written in the suggestive form

$$A_n(x) = (x + a)^n,$$

where it is understood that after expansion of the right member, a^k is replaced by a_k.

It also follows at once from (4.2) that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} x^k A_{n-k}(x) = a_n . \tag{4.4}$$

We may view (4.3) and (4.4) as an instance of the equivalence of (2.5) and (2.6).

If $a_0 \neq 0$, we may define the sequence $\{b_n\}$ by means of

$$\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} = \begin{cases} 1 & (n = 0) \\ 0 & (n > 0) \end{cases} , \tag{4.5}$$

or equivalently $A(z)B(z) = 1$, where

$$B(z) = \sum_{n=0}^{\infty} b_n z^n/n! \quad .$$

It then follows from (4.2) and (4.5) that

$$x^n = \sum_{k=0}^{n} \binom{n}{k} b_k A_{n-k}(x) \quad . \tag{4.6}$$

As an illustration we take the Bernoulli polynomial $B_n(x)$ defined by

$$\frac{ze^{xz}}{e^z-1} = \sum_{n=0}^{\infty} B_n(x) z^n/n! \quad ; \tag{4.7}$$

the Bernoulli number $B_n = B_n(0)$ is defined by

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n z^n/n! \quad . \tag{4.8}$$

Since

$$\frac{e^z - 1}{z} = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!} ,$$

it follows that

$$x^n = \sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} B_{n-k}(x) . \tag{4.9}$$

By means of (4.7) we can easily obtain the following basic properties of $B_n(x)$.

$$B_n(x+1) - B_n(x) = nx^n , \tag{4.10}$$

$$B_n(1-x) = (-1)^n B_n(x) , \tag{4.11}$$

$$\sum_{s=0}^{k-1} B_n\left(x + \frac{s}{k}\right) = k^{1-n} B_n(kx) \quad (k = 1,2,3,\cdots) . \tag{4.12}$$

Closely related to $B_n(x)$ is the Euler polynomial $E_n(x)$ defined by

$$\frac{2e^{xz}}{2^z + 1} = \sum_{n=0}^{\infty} E_n(x)\, z^n/n! . \tag{4.13}$$

Corresponding to (4.10), (4.11), (4.12) we have

$$E_n(x+1) + E_n(x) = 2x^n , \tag{4.14}$$

$$E_n(1-x) = (-1)^n E_n(x) , \tag{4.15}$$

$$\sum_{s=0}^{k-1} (-1)^s E_n\left(x + \frac{s}{k}\right) = k^{-n} E_n(kx) \qquad (k \text{ odd}) , \tag{4.16}$$

$$(4.17)\quad \sum_{s=0}^{k-1} (-1)^s B_{n+1}\left(x + \frac{s}{k}\right) = -\frac{n+1}{2k^n} E_n(kx) \qquad (k \text{ even}) .$$

For further developments the reader is referred to [14, Ch. 2].

5. Another important Appell set is furnished by the Hermite polynomials which may be defined by

$$(5.1)\quad e^{2xz-z^2} = \sum_{n=0}^{\infty} H_n(x)\, z^n/n! \quad .$$

Differentiating with respect to x we get

$$(5.2)\quad H_n'(x) = 2n\, H_{n-1}(x)$$

so that the definition (4.1) is modified slightly. If we differentiate (5.1) with respect to z we get

$$\sum_{n=0}^{\infty} H_{n+1}(x) z^n/n! = 2(x - z)e^{2xz-z^2} \quad ,$$

so that

$$(5.3)\quad H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \qquad (n \geq 1) .$$

Also, multiplying (5.1) by e^{z^2}, we get

$$(5.4)\quad (2x)^n = \sum_{2k\leq n} \frac{n!}{k!\,(n-2k)!} H_{n-2k}(x) \quad .$$

In the next place it follows from (5.1) that

$$\sum_{m,n=0}^{\infty} H_m(x)\, H_n(x) \frac{u^m v^n}{m!\, n!} = e^{2x(u+v)-u^2-v^2} = e^{2x(u+v)-(u+v)^2} e^{2uv}$$

$$= e^{2uv} \sum_{n=0}^{\infty} H_n(x)(u+v)^n/n! = \sum_{k=0}^{\infty} \frac{(2uv)^k}{k!} \sum_{m,n=0}^{\infty} H_{m+n}(x) \frac{u^m v^n}{m!\, n!} .$$

Equating coefficients we get

$$H_m(x)H_n(x) = \sum_{k=0}^{\min(m,n)} 2^k k! \binom{m}{k}\binom{n}{k} H_{m+n-2k}(x) . \tag{5.5}$$

Similarl y we have the inverse formula

$$H_{m+n}(x) = \sum_{k=0}^{\min(m,n)} (-1)^k 2^k k! \binom{m}{k}\binom{n}{k} H_{m-k}(x)H_{n-k}(x) . \tag{5.6}$$

The formulas (5.5), (5.6) are due to Nielsen [13]; (5.5) was rediscovered by Feldheim [8]. The above proof is due to Watson [20].

Another interesting formula is

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\, z^n/n! = (1 - 4z^2)^{-\frac{1}{2}} \exp\left\{\frac{4xyz - 4(x^2 + y^2)z^2}{1 - 4z^2}\right\} \tag{5.7}$$

We note first that

$$\begin{aligned} \sum_{n,k=0}^{\infty} H_{n+k}(x) \frac{z^n t^k}{n!\,k!} &= \sum_{n=0}^{\infty} H_n(x) \frac{(z + t)^n}{n!} \\ &= e^{2x(z+t)-(z+t)^2} \\ &= e^{2xz-z^2} e^{2(x-z)t-t^2} \\ &= e^{2xz-z^2} \sum_{k=0}^{\infty} H_k(x-z) \frac{t^k}{k!} . \end{aligned}$$

Equating coefficients, we get

$$\sum_{n=0}^{\infty} H_{n+k}(x)z^n/n! = e^{2xz-z^2} H_k(x - z) , \tag{5.8}$$

which reduces to (5.1) when $k = 0$.

Since, by (5.1),

$$H_n(x) = \sum_{2k \le n} (-1)^k \frac{n!}{k!\,(n-2k)!} (2x)^{n-2k} , \tag{5.9}$$

we have

$$\sum_{n=0}^{\infty} H_n(x) H_n(y) z^n/n!$$

$$= \sum_{n=0}^{\infty} \sum_{2k \le n} (-1)^k \frac{(2x)^{n-2k}}{k!\,(n-2k)!} H_n(y)\, z^n$$

$$= \sum_{n,k=0}^{\infty} (-1)^k \frac{(2x)^n z^{n+2k}}{k!\, n!} H_{n+2k}(y)$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{n!} \sum_{n=0}^{\infty} H_{n+2k}(y) \frac{(2xz)^n}{n!}$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{k!} e^{4xyz-4x^2z^2} H_{2k}(y - 2xz)$$

$$= e^{4xyz-4x^2z^2} \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{k!} \sum_{s=0}^{k} (-1)^s \frac{(2k)!}{s!\,(2k-2s)!} (2y - 4xz)^{2k-2s}$$

$$= e^{4xyz-4x^2z^2} \sum_{k,s=0}^{\infty} (-1)^k \frac{(2k+2s)!}{s!\,(2k)!\,(k+s)!} z^{2k+2s} (2y - 4xz)^{2k} .$$

Since

$$(2k)! = 2^{2k} k!\, (\tfrac{1}{2})_k ,$$

we get

$$e^{4xyz-4x^2z^2} \sum_{k,s=0}^{\infty} (-1)^k \frac{(\frac{1}{2})_{k+s}}{s!\,k!\,(\frac{1}{2})_k} 2^{2s} z^{2k+2s} (2y - 4xz)^{2k}$$

$$= e^{4xyz-4x^2z^2} \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}(2y - 4xz)^{2k}}{k!} \sum_{s=0}^{\infty} \frac{(k+\frac{1}{2})_s}{s!} (2z)^{2s}$$

$$= e^{4xyz-4x^2z^2} \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}(2y - 4xz)^{2k}}{k!} (1 - 4z^2)^{-k-\frac{1}{2}}$$

$$= (1 - 4x^2)^{-\frac{1}{2}} \exp\left\{4xyz - 4x^2z^2 - \frac{z^2(2y - 4xz)^2}{1 - 4z^2}\right\} .$$

$$= (1 - 4x^2)^{-\frac{1}{2}} \exp\left\{\frac{4xyz - 4(x^2 + y^2)z^2}{1 - 4z^2}\right\} .$$

This completes the proof of (5.7). The proof is taken from Rainville [16, p. 197].

6. The formula of Saalschutz [1, p. 9],

$$\sum_{k=0}^{n} \frac{(-n)_k (a)_k (b)_k}{k!\,(c)_k (d)_k} = \frac{(c - a)_n (c - b)_n}{(c)_n (c - a - b)_n} , \tag{6.1}$$

where

$$c + d = -n + a + b + 1, \tag{6.2}$$

is very useful in many instances.

If we replace c by $c - n$, (6.1) becomes

$$\sum_{k=0}^{n} \frac{(-n)_k (a)_k (b)_k}{k!\,(c-n)_k (d)_k} = \frac{(d - a)_n (d - b)_n}{(d)_n (d - a - b)_n} , \tag{6.3}$$

where now

$$c + d = a + b + 1 . \tag{6.4}$$

Now by (6.3)

$$\sum_{n=0}^{\infty} \frac{(d-a)_n (d-b)_n}{x!\,(d)_n} x^n = \sum_{n=0}^{\infty} \frac{(d-a-b)_n}{n!} x^n \sum_{k=0}^{n} \frac{(-n)_k (a)_k (b)_k}{k!\,(c-n)_k (d)_k}$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k (b)_k}{k!\,(d)_k} x^k \sum_{n=0}^{\infty} \frac{(d-a-b)_n}{n!} x^n$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k (b)_k}{k!\,(d)_k} x^k (1-x)^{a+b-d} .$$

Thus (6.3) is equivalent to

$$(6.5) \qquad F(a,b;\, d;\, x) = (1-x)^{d-a-b} F(d-a,\, d-b;\, d;\, x) ,$$

where $F(a,b;\, d;\, x)$ denotes the hypergeometric function.

It is customary to prove (6.5) by making use of the differential equation of the second order satisfied by $F(a,b;c;x)$. We shall, however, give an inductive proof of (6.1) which we now write in the form

$$(6.6) \qquad \sum_{k=0}^{n} \frac{(-n)_k (a+n)_k (b)_k}{k!\,(c)_k (d)_k} = \frac{(c-b)_n (d-b)_n}{(c)_n (d)_n} ,$$

where

$$(6.7) \qquad c + d = a + b + 1 .$$

Let

$$S_n(a,b,c,d) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(a+n)_k (b)_k}{(c)_k (d)_k} ,$$

where a,b,c,d satisfy (6.7). Then

$$S_{n+1}(a,b,c,d) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(a+n+1)_k (b)_k}{(c)_k (d)_k} - \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(a+n+1)_{k+1} (b)_{k+1}}{(c)_{k+1} (d)_{k+1}}$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(a+n+1)_k (b)_k}{(c)_{k+1} (d)_{k+1}} \left\{(c+k)(d+k) - (a+n+k+1)(b+k)\right\}.$$

Now put

$$(c+k)(d+k) - (a+n+k+1)(b+k) = A(d+k) + B(c+k),$$

where A, B are independent of k. Then

$$\text{(6.8)} \qquad \begin{cases} (d-c)A = (c-b)(a-c+n+1), \\ (c-d)B = (d-b)(a-d+n+1). \end{cases}$$

It follows that

$$S_{n+1}(a,b,c,d) = \frac{A}{c} S_n(a+1,b,c+1,d) + \frac{B}{d} S_n(a+1,b,c,d+1).$$

Assuming that (6.6) holds, we therefore get

$$S_{n+1}(a,b,c,d) = \frac{A}{c} \frac{(c-b+1)_n (d-b)_n}{(c+1)_n (d)_n} + \frac{B}{d} \frac{(c-b)_n (d-b+1)_n}{(c)_n (d+1)_n}$$

$$= \frac{(c-b+1)_{n-1} (d-b+1)_{n-1}}{(c)_{n+1} (d)_{n+1}} \left\{A(d-b)(c-b+n)(d+n) + B(c-b)(d-b+n)(c+n)\right\}$$

By (6.8),

$$\begin{aligned} &(d-c)\left\{A(d-b)(c-b+n)(d+n) + B(c-b)(d-b+n)(c+n)\right\} \\ &\quad = (c-b)(d-b)\left\{(c-b+n)(d+n)(a-c+n+1) - (d-b+n)(c+n)(a-d+n+1)\right\} \\ &\quad = (c-b)(d-b)\left\{(c-b+n)(d+n)(d-b+n) - (d-b+n)(c+n)(c-b+n)\right\} \\ &\quad = (c-b)(d-b)(c-b+n)(d-b+n)(d-c). \end{aligned}$$

Therefore

$$S_{n+1}(a,b,c,d) = \frac{(c-b)_{n+1}(d-b)_{n+1}}{(c)_{n+1}(d)_{n+1}} ,$$

which completes the induction.

As an application we take (6.6) in the form

$$\sum_{j=0}^{k} \frac{(-k)_j(a+k)_j(-a+b+c+1)_j}{j!\,(b+1)_j(c+1)_j} = \frac{(a-b)_k(a-c)_k}{(b+1)_k(c+1)_k} , \tag{6.9}$$

where now a,b,c are arbitrary. Then

$$\sum_{k=0}^{\infty} \frac{(a)_k(a-b)_k(a-c)_k}{k!\,(b+1)_k(c+1)_k}\, x^k$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{j=0}^{k} \frac{(-k)_j(a)_{j+k}(-a+b+c+1)_j}{j!\,(b+1)_j(c+1)_j}$$

$$= \sum_{j=0}^{\infty} (-1)^j \frac{(a)_{2j}(-a+b+c+1)_j}{j!\,(b+1)_j(c+1)_j}\, x^j \sum_{k=0}^{\infty} \frac{(a+2j)_k}{k!}\, x^k ,$$

so that we have

$$\sum_{k=0}^{\infty} \frac{(a)_k(a-b)_k(a-c)_k}{k!\,(b+1)_k(c+1)_k}\, x^k = \sum_{j=0}^{\infty} (-1)^j \frac{(a)_{2j}(-a+b+c+1)_j}{j!\,(b+1)_j(c+1)_j}\, x^j(1-x)^{-a-2} \tag{6.10}$$

If we take $a = -2n$, $x = 1$; (6.10) reduces to

$$\sum_{k=0}^{2n} \frac{(-2n)_k(-2n-b)_k(-2n-c)_k}{k!\,(b+1)_k(c+1)_k} = (-1)^n \frac{(2n)!\,(b+c+2n+1)_n}{n!\,(b+1)_n(c+1)_n} . \tag{6.11}$$

In particular, for $b = c = 0$, (6.11) becomes Dixon's theorem:

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)\, \frac{(3n)!}{(n!)^3} . \tag{6.12}$$

Note also that (6.10) implies, for $a = -n$, $b = c = 0$,

$$\sum_{k=0}^{n} \binom{n}{k}^3 x^k = \sum_{2j \le n} \frac{(n+j)!}{(j!)^3(n-j)!} x^j(1+x)^{n-2j} , \tag{6.13}$$

and in particular

$$\sum_{k=0}^{n} \binom{n}{k}^3 = \sum_{2j \le n} \frac{(n+j)!}{(j!)^3(n-j)!} 2^{n-2j} , \tag{6.14}$$

a result due to MacMahon. For other proofs of these formulas see [17, pp. 41, 42].

7. We now turn to some problems involving multiple generating functions. To begin with, we take

$$\begin{aligned}
(1 - 2x - 2y + x^2 - 2xy + y^2)^{-\frac{1}{2}} &= \left[(1-x-y)^2 - 4xy\right]^{-\frac{1}{2}} \\
&= (1-x-y)^{-1}\left[1 - \frac{4xy}{(1-x-y)^2}\right]^{-\frac{1}{2}} \\
&= \sum_{r=0}^{\infty} \binom{2r}{r} \frac{(xy)^r}{(1-x-y)^{2r+1}} \\
&= \sum_{r=0}^{\infty} \binom{2r}{r} (xy)^r \sum_{s,t=0}^{\infty} \frac{(2r+s+t)!}{(2r)!\,s!\,t!} x^s y^t \\
&= \sum_{m,n=0}^{\infty} x^m y^n \sum_{r=0}^{\min(m,n)} \frac{(m+n)!}{r!\,r!\,(m-r)!\,(n-r)!} .
\end{aligned}$$

Since

$$\sum_{r=0}^{\min(m,n)} \frac{(m+n)!}{r!\,r!\,(m-r)!\,(n-r)!} = \binom{m+n}{m} \sum_{r=0}^{\min(m,n)} \binom{m}{r}\binom{n}{r} = \binom{m+n}{m}^2 ,$$

we have

$$(7.1)\qquad (1 - 2x - 2y + x^2 - 2xy + y^2)^{-\frac{1}{2}} = \sum_{m,n=0}^{\infty} \binom{m+n}{m}^2 x^m y^n .$$

This is in fact a disguised form of the generating function for Legendre polynomials:

$$(7.2)\qquad (1 - 2xz + z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)\, z^n .$$

However to save space, we shall not elaborate this point.

One can extend (7.1) in various ways. For example, we can construct the generating function for the Jacobi polynomial

$$(7.3)\qquad P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k}\binom{n+\beta}{k}\left(\frac{x-1}{2}\right)^k\left(\frac{x+1}{2}\right)^{n-k} .$$

It is known that

$$(7.4)\qquad \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x) z^n = 2^{\alpha+\beta} R^{-1}(1 - z + r)^{-\alpha}(1 - z + R)^{-\beta} ,$$

where

$$R = (1 - 2xz + z^2)^{\frac{1}{2}} .$$

For a proof of (7.4) see, for example, [16, p. 140].

If we put

$$u = \tfrac{1}{2}(x - 1)z, \quad v = \tfrac{1}{2}(x + 1)z ,$$

we have

$$(7.5)\qquad R = [(1 - u - v)^2 - 4uv]^{\frac{1}{2}}$$

and (7.4) becomes

$$(7.5)\qquad \sum_{j,k=0}^{\infty} \binom{\alpha + j + k}{j}\binom{\beta + j + k}{k} u^j v^k = 2^{\alpha+\beta} R^{-1}(1 - u + v + R)^{-\alpha}(1 + u - v + R)^{-\beta}$$

with R defined by (7.5).

We shall now give a simple proof of (7.6). Consider the expression

$$(1 - x)^{-\alpha-1}(1 - y)^{-\beta-1} \sum_{j,k=0}^{\infty} \frac{(\alpha + 1)_{j+k}(\beta + 1)_{j+k}}{j!\,k!\,(\alpha + 1)_k(\beta + 1)_j} \frac{(-1)^{j+k}x^j y^k}{(1 - x)^{j+k}(1 - y)^{j+k}}$$

$$= \sum_{j,k=0}^{\infty} (-1)^{j+k} \frac{(\alpha+1)_{j+k}(\beta+1)_{j+k}}{j!\,k!\,(\alpha+1)_k(\beta+1)_j} x^j y^k \sum_{r,s=0}^{\infty} \frac{(\alpha+j+k+1)_r(\beta+j+k+1)_s}{r!\,s!} x^r y^s$$

$$= \sum_{m,n=0}^{\infty} (\alpha+1)_m(\beta+1)_n x^m y^n \sum_{j=0}^{m}\sum_{k=0}^{n} \frac{(-m)_j(-n)_k}{j!\,k!} \frac{(\alpha+m+1)_k(\beta+n+1)_j}{(\alpha + 1)_k(\beta + 1)_j} .$$

The inner sum is equal to

$$\sum_{j=0}^{m} \frac{(-m)_j}{j!} \frac{(\beta + n + 1)_j}{(\beta + 1)_j} \sum_{k=0}^{k} \frac{(-n)_k(\alpha + m + 1)_k}{k!\,(\alpha + 1)_k} = \frac{(-n)_m}{(\beta + 1)_m} \frac{(-m)_n}{(\alpha + 1)_n}$$

by (2.3), which vanishes unless m = n. It follows that

$$(7.7)\qquad (1 - x)^{-\alpha-1}(1 - y)^{-\beta-1} \sum_{j,k=0}^{\infty} \binom{\alpha + j + k}{j}\binom{\beta + j + k}{k} \frac{(-1)^{j+k}x^j y^k}{(1 - x)^{j+k}(j - y)^{j+k}} = \frac{1}{1-xy} .$$

Now put

$$u = - \frac{x}{(1 - x)(1 - y)}, \quad v = \frac{y}{(1 - x)(1 - y)} .$$

Then

$$1 - x = \frac{2}{1 - u + v + R}, \quad 1 - y = \frac{2}{1 + u - v + R}, \quad \frac{1 - xy}{(1 - x)(1 - y)} = R$$

and (7.7) reduces to (7.6).

8. We shall now extend (7.1) in another direction, namely a larger number of variables. Consider first

$$[(1 - x - y - z)^2 - 4xyz]^{-\frac{1}{2}} = \sum_{r=0}^{\infty} \binom{2r}{r} \frac{(xyz)^r}{(1 = x - y - z)^{2r+1}} .$$

Since

$$(1 - x - y - z)^{-2r-1} = \sum_{k=0}^{\infty} \binom{2r+k}{k} (x + y + z)^k$$

$$= \sum_{s,t,u=0}^{\infty} \frac{(2r+s+t+u)!}{(2r)!\, s!\, t!\, u!} x^s y^t z^u ,$$

we get

$$[(1 - x - y - z)^2 - 4xyz]^{-\frac{1}{2}} = \sum_{r=0}^{\infty} \binom{2r}{r} (xyz)^r \sum_{s,t,u=0}^{\infty} \frac{(2r + s + t + u)!}{(2r)!\, s!\, t!\, u!} x^s y^t z^u$$

$$= \sum_{m,n,p=1}^{\infty} x^m y^y z^p \sum_{r=0}^{\min(m,n,p)} \frac{(m+n+p-r)!}{r!\,r!\,(m-r)!\,(n-r)!\,(p-r)!}$$

Now by (6.1)

$$\sum_{r=0}^{\min(m,n,p)} \frac{(m + n + p - r)!}{r!\,r!\,(m - r)!\,(n - r)!\,(p - r)!} = \frac{(m + n + p)!}{m!\,n!\,p!} \sum_{r=0}^{-m} \frac{(-m)_r(-n)_r(-p)_r}{r!\,r!\,(-m - n - p)_r}$$

$$= \frac{(m + n + p)!}{m!\,n!\,p!} \frac{(n + 1)_m (p + 1)_m}{m!\,(n + p + 1)_m}$$

$$= \frac{(m + n)!\,(m + p)!\,(n + p)!}{m!\,m!\,n!\,n!\,p!\,p!}$$

$$= \binom{m+n}{m}\binom{n+p}{n}\binom{p+m}{p} .$$

Finally therefore we have

$$(8.1)\quad [(1-x-y-z)^2-4xyz]^{-\frac{1}{2}} = \sum_{m,n,p=0}^{\infty} \binom{m+n}{m}\binom{n+p}{n}\binom{p+m}{p} x^m y^n z^p .$$

To carry this further a different approach seems necessary. In the expansion

$$(1-v)^{-1-i} = \sum_{j=0}^{\infty} \binom{i+j}{j} v^j$$

replace v by $v/(1-w)$ and multiply by $(1-w)^{-1}$. Then

$$\frac{(1-w)^i}{(1-v-w)^{i+1}} = \sum_{j=0}^{\infty} \binom{i+j}{j} v^j (1-w)^{-j-1}$$

$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \binom{i+j}{j}\binom{j+k}{k} v^j w^k .$$

Next replacing w by $w/(1-x)$, we get

$$\frac{(1-w-x)^i}{[(1-v)(1-x)-w]^{i+1}} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty} \binom{i+j}{j}\binom{j+k}{k}\binom{k+r}{r} v^j w^k x^r.$$

Now replace x by $x/(1-y)$. This yields

$$(8.2)\quad \frac{[(1-w)(1-y)-x]^i}{[(1-v)(1-x-y)-(1-y)w]^{i+1}} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \cdot \binom{i+j}{j}\binom{j+k}{k}\binom{k+r}{r}\binom{r+s}{s} v^j w^k x^r y^s .$$

Now multiply both sides of (8.2) by $u^i y^{-i}$ and sum over i. It follows that

$$(8.3)\qquad \left\{(1 - v)(1 - x - y) - (1 - y)w - [(1 - w)(1 - y) - x]uy^{-1}\right\}^{-1}$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\binom{i+j}{j}\binom{j+k}{k}\binom{k+r}{r}\binom{r+s}{s}u^i v^j w^k x^r y^{s-i} .$$

We are concerned with that part of the multiple sum that is independent of y. The left member of (8.3) is equal to

$$\left\{[(1 - v)(1 - x) - w + u(1 - w)] - (1 - v - w)y - (1 - w - x)xy^{-1}\right\}^{-1}$$

$$= \sum_{r=0}^{\infty}\frac{[(1 - v - w)y + (1 - w - x)uy^{-1}]^r}{[(1 - v)(1 - x) - w + u(1 - w)]^{r+1}} .$$

Expanding the numerator by the binomial theorem, it is clear that the terms independent of y contribute

$$\sum_{r=0}^{\infty}\binom{2r}{r}\frac{(1 - v - w)^r(1 - w - x)^r u^r}{[(1 - v)(1 - x) - w + u(1 - w)]^{2r+1}}$$

$$= \left\{[(1 - v)(1 - x) - w + u(1 - w)]^2 - 4u(1 - v - w)(1 - w - x)\right\}^{-\frac{1}{2}}$$

$$= \left\{(1 - u - v - w - x + uw + vx)^2 - 4uvwx\right\}^{-\frac{1}{2}}$$

We have therefore proved

$$(8.4)\qquad \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\binom{i+j}{j}\binom{j+k}{k}\binom{k+r}{r}\binom{r+i}{i}u^i v^j w^k x^r$$

$$= \left\{(1 - u - v - w - x + uw + vx)^2 - 4uvwx\right\}^{-\frac{1}{2}} .$$

We now specialize (8.4) by taking $u = w,\ v = x$. Since

$$(1 - 2u - 2w + u^2 + w^2)^2 - 4u^2w^2 = (1 - u - v)^2(1 - 2u - 2v + u^2 - 2uv + v^2)$$

$$= (1 - u - v)^2[(1 - u - v)^2 - 4uv],$$

Eq. (8.4) becomes

$$(8.5)\qquad \sum_{m,n=0} H(m,n)u^m v^n = (1-u-v)^{-1}[(1-u-v)^2 - 4uv]^{-\frac{1}{2}}$$

where

$$(8.6)\qquad H(m,n) = \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{i+j}{j}\binom{m-i+j}{j}\binom{i+n-j}{n-j}\binom{m-i+n-j}{n-j}.$$

If we multiply (8.5) by $1 - u - v$ and apply (7.1), we get

$$(8.7)\qquad H(m,n) - H(m-1,n) - H(m,n-1) = \binom{m+n}{m}^2 ,$$

an identity due to Paul Brock [2], [3]. We remark also that (8.5) implies

$$(8.8)\qquad H(m,n) = \sum_{r=0}^{m}\sum_{s=0}^{n}\binom{r+s}{s}^2\binom{m-r+n-s}{m-r} .$$

Also, since

$$\begin{aligned}(1-u-v)^{-1}[(1-u-v)^2 - 4uv]^{-\frac{1}{2}} &= \sum_{r=0}^{\infty}\binom{2r}{r}(uv)^r(1-u-v)^{-2r-2}\\ &= \sum_{r=0}^{\infty}\binom{2r}{r}(uv)^r\sum_{s,t=0}^{\infty}\frac{(2r+s+t+1)!}{(2r+1)!\,s!\,t!}u^s v^t\\ &= \sum_{m,n=0}^{\infty}u^m v^n\sum_{r=0}^{\min(m,n)}\binom{2r}{r}\frac{(m+n+1)!}{(2r+1)!(m-r)!(n-r)!}\end{aligned}$$

it follows that

$$(8.9)\qquad H(m,n) = \binom{m+n}{m}\sum_{r=0}^{\min(m,n)}\frac{m+n+1}{2r+1}\binom{m}{r}\binom{n}{r}.$$

For the generalized version of (8.4), see [4], [6], [18, Ch. 4].

9. We shall now briefly discuss some enumerative problems. The problem of permutations with a given number of inversions was called to the writer's attention by H. W. Gould. Let $\{a_1, a_2, \cdots, a_n\}$ denote a permutation of $\{1, 2, \cdots, n\}$. The pair a_i, a_j is called an <u>inversion</u> provided that $i < j$ but $a_i > a_j$. Thus $\{1, 2, \cdots, n\}$ has no inversions, while $\{n, n-1, \cdots, 1\}$ has $n(n-1)/2$ inversions. Let $B(n, r)$ denote the number of permutations of $\{1, 2, \cdots, n\}$ with r inversions. Clearly, $0 \leq r \leq n(n-1)/2$.

From the definition, it follows that

$$B(n+1, r) = \sum_{\substack{s=0 \\ s \leq n}}^{r} B(n, r-s) . \tag{9.1}$$

This recurrence is obtained when the element $n+1$ is adjoined to any permutation of $\{1, 2, \cdots, n\}$. Now put

$$\beta_n(x) = \sum_{r=0}^{n(n-1)/2} B(n, r)x^r .$$

Then by (9.1),

$$\begin{aligned} \beta_{n+1}(x) &= \sum_{r=0}^{n(n+1)/2} x^r \sum_{\substack{s=0 \\ s \leq n}}^{r} B(n, r-s) \\ &= \sum_{s=0}^{n} x^s \sum_{r=0}^{n(n-1)/2} B(n, r)x^r , \end{aligned}$$

so that

$$\beta_{n+1}(x) = (1 + x + \cdots + x^n)\beta_n(x) . \tag{9.2}$$

Since $\beta_1(x) = 1$, (9.2) yields

$$\beta_n(x) = \frac{(1-x)(1-x^2) \cdots (1-x^n)}{(1-x)^n} . \tag{9.3}$$

Thus, for example,

$$B(n,0) = 1, \quad B(n,1) = n - 1, \quad B(n,2) = \tfrac{1}{2}(n+1)(n-2) \qquad (n > 1),$$

$$B(n,3) = \tfrac{1}{6}\, n(n^2 - 7) \qquad (n > 2),$$

$$B(n,4) = \tfrac{1}{24}\, n(n+1)(n^2 - n - 14) \qquad (n > 3).$$

From (9.3), we get the generating function

$$\sum_{n=0}^{\infty} \beta_n(x) z^n/(x)_n = \frac{1 - x}{1 - x - z}, \tag{9.4}$$

where

$$(x)_n = (1 - x)(1 - x^2) \cdots (1 - x^n), \qquad (x)_0 = 1.$$

This is the first occurrence in the present paper of a generating function with denominator $(x)_n$; see the remark in Section 11 below.

If we make use of Euler's formula

$$\begin{aligned} \prod_{n=1}^{\infty} (1 - x^n) &= \sum_{k=-\infty}^{\infty} (-1)^k x^{\frac{1}{2}k(3k+1)} \\ &= 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \cdots, \end{aligned} \tag{9.5}$$

we obtain an explicit formula for $B(n,r)$ when $r \le n$. For example, we have

$$B(n,4) = \binom{n+3}{4} - \binom{n+2}{3} - \binom{n+1}{2} \qquad (n \ge 4),$$

$$B(n,5) = \binom{n+4}{5} - \binom{n+3}{4} - \binom{n+2}{3} + 1 \qquad (n \ge 5),$$

$$B(n,6) = \binom{n+5}{6} - \binom{n+4}{5} - \binom{n+3}{4} + n \qquad (n \ge 6).$$

If we rewrite (9.3) in the form

$$\beta_n(x) = (1 + x)(1 + x + x^2) \cdots (1 + x + \cdots + x^{n-1}) ,$$

we obtain the following combinatorial theorem: $B(n,r)$ is equal to the number of (integral) solutions $x_1, x_2, \cdots, x_n$ of the equation

$$x_1 + x_2 + \cdots + x_n = r \tag{9.6}$$

subject to the conditions

$$0 \leq x_k < k \qquad (k = 1, 2, \cdots, n) .$$

We remark also that (9.3) implies

$$\sum_{r=0}^{n(n-1)/2} B(n,r) = n! \quad ,$$

$$\sum_{r=0}^{n(n-1)/2} (-1)^r B(n,r) = 0 \qquad (n > 1) ,$$

$$\sum_{r=0}^{n(n-1)/2} r\, B(n,r) = n! \sum_{k=1}^{n} \frac{1}{k} \binom{k}{2} = \frac{1}{4} n(n-1) \cdot n! \quad .$$

For references, see [12, pp. 94-97].

10. As a second enumerative problem, we consider permutations with a given number of rises. If $\{a_1, a_2, \cdots, a_n\}$ is a permutation of $\{1, 2, \cdots, n\}$, a_j, a_{j+1} is a rise provided $a_j < a_{j+1}$. By convention there is always a rise preceding a_1. For example, the permutation $\{3,4,1,2\}$ has 3 rises.

Let $A_{n,k}$ denote the number of permutations of $\{1, 2, \cdots, n\}$ with k rises. Then we have the recurrence

$$A_{n+1,k} = (n - k + 2)A_{n,k-1} + kA_{n,k} \, . \tag{10.1}$$

The proof is simple. Let $\{a_1, \cdots, a_n\}$ be a permutation of $\{1, 2, \cdots, n\}$. If $a_i < a_{i+1}$ and we place $n + 1$ between a_i and a_{i+1} the number of rises is unchanged. If, however, $a_i > a_{i+1}$, the number of rises is increased by 1; this is also true when $n + 1$ is placed to the right of a_n.

It is also clear from the definition that

$$A_{n,1} = A_{n,n} = 1 \quad (n = 1,2,3,\cdots); \tag{10.2}$$

the permutations in question are $\{n, n-1, \cdots, 1\}$ and $\{1,2,\cdots,n\}$, respectively. By means of (10.1) and (10.2), we can easily compute the first few values of $A_{n,k}$.

$$\begin{array}{lllll}
1 & & & & \\
1 & 1 & & & \\
1 & 4 & 1 & & \\
1 & 11 & 11 & 1 & \\
1 & 26 & 66 & 26 & 1
\end{array}$$

If, in a given permutation $\{a_1,a_2,\cdots,a_k\}$, we replace a_k by $n - a_k + 1$ $(k = 1,2,\cdots,n)$, it follows that

$$A_{n,k} = A_{n,n-k+1}. \tag{10.3}$$

Also it is evident that

$$\sum_{k=1}^{n} A_{n,k} = n! \quad . \tag{10.4}$$

Put

$$A_0(x) = 1, \quad A_n(x) = \sum_{k=1}^{n} A_{n,k}\, x^{k-1} \quad (n = 1,2,3,\cdots).$$

Then it can be shown that

$$\frac{1 - x}{e^z - x} = \sum_{n=0}^{\infty} (x - 1)^{-n} A_n(x)\, z^n/n! \tag{10.5}$$

We shall not give the proof of (10.5). It is indeed easier to define $A_n(x)$ by means of (10.5) and show that the other properties follow from this definition.

For references, see [5], [18, Ch. 8].

The symmetry property (10.3) is not obvious from (10.5). This suggests the following change in notation. Put

$$A(r,s) = A_{r+s+1,\, r+1} . \tag{10.6}$$

Then by (10.3),

$$A(r,s) = A(s,r) . \tag{10.7}$$

Also (10.5) implies, after a little manipulation,

$$F(x,y) = \frac{e^x - e^y}{xe^y - ye^x} = \sum_{r,s=0}^{\infty} A(r,s) \frac{x^r y^s}{(r+s+1)!} . \tag{10.8}$$

Another symmetrical generating function is

$$(1 + xF(x,y))(1 + xF(x,y)) = \sum_{r,s=0}^{\infty} A(r,s) \frac{x^r y^s}{(r+s)!} . \tag{10.9}$$

The denominator in the right members of (10.8) and (10.9) should be noticed.

11. We conclude with a few remarks about q-series; an instance has appeared in (9.4). Simple examples are

$$\prod_{n=0}^{\infty} (1 - x^n s)^{-1} = \sum_{n=0}^{\infty} z^n/(x)_n , \tag{11.1}$$

$$\prod_{n=0}^{\infty} (1 + x^n z) = \sum_{n=0}^{\infty} x^{\frac{1}{2}n(n-1)} z^n/(x)_n , \tag{11.2}$$

where as above

$$(x)_0 = 1, \quad (x)_n = (1 - x)(1 - x^2) \cdots (1 - x^n) .$$

A more general result that includes both (11.1) and (11.2) is

$$\prod_{n=0}^{\infty} \frac{1 - ax^n z}{1 - x^n z} = \sum_{n=0}^{\infty} \frac{(a)_n}{(x)_n} z^n , \tag{11.3}$$

where

$$(a)_0 = 1, \quad (a)_n = (1 - a)(1 - ax) \cdots (1 - ax^{n-1}) .$$

To prove (11.3), put

$$F(z) = \prod_{n=0}^{\infty} \frac{1 - ax^n z}{1 - x^n z} = \sum_{n=0}^{\infty} A_n z^n ,$$

where A_n is independent of z. Then

$$F(xz) = \frac{1 - z}{1 - az} F(z) ,$$

so that

$$(1 - az) \sum_{n=0}^{\infty} A_n x^n z^n = (1 - z) \sum_{n=0}^{\infty} A_n z^n .$$

This gives

$$(1 - x^n)A_n = (1 - ax^{n-1})A_{n-1} ,$$

and (11.3) follows at once.

In particular, for $a = x^k$, (11.3) becomes

$$(11.4)\qquad \prod_{n=0}^{k-1}(1 - x^n z)^{-1} = \sum_{n=0}^{\infty}\frac{(x^k)_n}{(x)_n} z^n = \sum_{n=0}^{\infty}\begin{bmatrix} k+n-1 \\ n \end{bmatrix} z^n ,$$

where

$$\begin{bmatrix} k \\ n \end{bmatrix} = \frac{(x)_k}{(x)_n (x)_{k-n}} \qquad (0 < n \le k) .$$

If we take $a = x^{-k}$ and replace z by $x^k z$ we get

$$(11.5)\qquad \prod_{n=0}^{k-1}(1 - x^n z) = \sum_{n=0}^{k}(-1)^n \begin{bmatrix} k \\ n \end{bmatrix} x^{\frac{1}{2}n(n-1)} z^n .$$

Note that when $x = 1$, $\begin{bmatrix} k \\ n \end{bmatrix}$ reduces to $\binom{k}{n}$.

It also follows from (11.3) that

$$(11.6)\qquad \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(a)_k (b)_{n-k} a^{n-k} = (ab)_n ,$$

for arbitrary a,b. Specializing a,b or using (11.5), we get

$$(11.7)\qquad \sum_{=0}^{k}\begin{bmatrix} m \\ k-s \end{bmatrix}\begin{bmatrix} n \\ s \end{bmatrix} x^{s^2-ks+ms} = \begin{bmatrix} m+n \\ k \end{bmatrix} .$$

which evidently generalizes (2.2).

The function

$$e(z) = \prod_{n=0}^{\infty}(1 - x^n z)^{-1}$$

can be thought of as an analog of the exponential function. This suggests the definition (compare (4.2)),

$$e(tz)\sum_{n=0}^{\infty} a_n z^n/(x)_n = \sum_{n=0}^{\infty} A_n(t)\, z^n/(x)_n \quad , \tag{11.8}$$

where a_n is a function of x that is independent of t and z. Using (11.1), we get

$$A_n(t) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} a_k\, t^{n-k} \quad . \tag{11.9}$$

If we define the operator Δ by means of

$$\Delta f(t) = f(t) - f(xt) \; ,$$

it follows at once from (11.9) that

$$\Delta A_n(t) = (1 - x^n)A_{n-1}(t) \quad . \tag{11.10}$$

Conversely if a set of polynomials in t satisfy (11.10), then there exists a sequence $\{a_n\}$ independent of t such that (11.8) holds.

The special case $a_n = 1$ is of particular interest. Put

$$e(tz)e(z) = \sum_{n=0}^{\infty} H_n(t)z^n/(x)_n \; ,$$

so that

$$H_n(t) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} t^k \, .$$

For properties of these and related polynomials, see [7], [11], [19]. The $H_n(t)$ are in some respects analogous to the Hermite polynomials. We cite the bilinear generating function

$$(11.11)\quad \sum_{n=0}^{\infty} H_n(u)H_n(v)z^n/(x)_n = \frac{e(z)e(uz)e(vz)e(uvz)}{e(uvz^2)}$$

$$= \prod_{n=0}^{\infty} \frac{1 - x^n uvz^2}{(1 - x^n z)(1 - x^n uz)(1 - x^n vz)(1 - x^n uvz)}$$

which may be compared with (5.7). For proof of (11.11), see [7].

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