ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-175 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Put

$$\left(1 + z + \frac{1}{3}z^2\right)^{-n-1} = \sum_{k=0}^{\infty} a(n,k)z^k .$$

Show that

$$\text{(I)} \qquad a(n,n) = \frac{2\cdot 5\cdot 8 \cdots (3n-1)}{n!}$$

$$\text{(II)} \qquad \sum_{s=0}^{n} \binom{n-s}{s}\binom{2n-s}{n}\left(-\frac{1}{3}\right)^s = \frac{2\cdot 5\cdot 8 \cdots (3n-1)}{n!}$$

$$\text{(III)} \qquad \sum_{r=0}^{\infty} \binom{n+r}{r}\binom{2n-r}{n}(-\omega)^r = (\omega^2\sqrt{-3})^n \frac{2\cdot 5\cdot 8 \cdots (3n-1)}{n!} ,$$

where

$$\omega = \frac{1}{2}(-1 + \sqrt{-3}) .$$

H-176 Proposed by C. C. Yalavigi, Government College, Mercara, India.

In the "Collected Papers of Srinivas Ramanujan," edited by G. H. Hardy, P. V. Sheshu Aiyer, and B. M. Wilson, Cambridge University Press, 1927, on p. 326, Q. 427 reads as follows:

Show that

$$\frac{1}{e^{2\pi} - 1} + \frac{2}{e^{4\pi} - 1} + \frac{3}{e^{6\pi} - 1} + \cdots = \frac{1}{24} + \frac{1}{8\pi} .$$

Provide a proof.

H-177 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Let $R(N)$ denote the number of solutions of

$$N = F_{k_1} + F_{k_2} + \cdots + F_{k_r} \qquad (r = 1, 2, 3, \cdots),$$

where

$$k_1 > k_2 > \cdots > k_r > 1 .$$

Show that

(1) $R(F_{2n}F_{2m}) = R_{2n+1}F_{2m}) = (n - m)F_{2m} + F_{2m-1}$ $\quad (n \geq m)$,

(2) $R(F_{2n}F_{2m+1}) = (n - m)F_{2m+1}$ $\quad (n > m)$,

(3) $R(F_{2n+1}F_{2m+1}) = (n - m)F_{2m+1}$ $\quad (n > m)$,

(4) $R(F^2_{2n+1}) = R(F^2_{2n}) = F_{2n-1}$ $\quad (n \geq 1)$.

SOLUTIONS

SUM INVERSION

H-151 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

a. Put

$$(1 - ax^2 - bxy - cy^2)^{-1} = \sum_{m,n=0}^{\infty} A_{m,n} x^m y^n .$$

Show that

$$\sum_{n=0}^{\infty} A_{n,n} x^n = \{1 - 2bx + (b^2 - 4ac)x^2\}^{-\frac{1}{2}} .$$

B. Put

$$(1 - ax - bxy - cy)^{-1} = \sum_{m,n=0}^{\infty} B_{m,n} x^m y^n .$$

Show that

$$\sum_{n=0}^{\infty} B_{n,n} x^n = \{(1 - bx)^2 - 4acx\}^{-\frac{1}{2}} .$$

Solution by M. L. J. Hautus and D. A. Klarner, Technological University, Eindhouen, the Netherlands.

In a paper submitted to the Duke Mathematical Journal (the diagonal of a double power series), we have proved the following result:

Theorem. Suppose

$$F(x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f(m,n)x^m y^n$$

converges for all x and y such that $|x| < A$, $|y| < B$, then for all z such that $|z| < AB$, we have

$$\frac{1}{2\pi i}\int_C F(s,z/s)\,\frac{ds}{s} = \sum_{n=0}^{\infty} f(n,n)z^n ,$$

where C is the circle $\{s:|s| = (A + |z|/B)/2\}$. Furthermore, if $F(s,z/s)/s$ has only isolated singularities inside C, then the integral can be evaluated by summing the residues of $F(s,z/s)/s$ at these singularities. Coincidentally, we gave Carlitz' Problem B as an example in our paper. Problem A can be treated in just the same way. According to the theorem cited,

$$\sum_{n=0}^{\infty} A_{n,n}\,x^n = \frac{1}{2\pi i}\int_C \frac{-s\,ds}{as^4 - (1 - bx)s^2 + cx^2}$$

The singularities of the integrand are

$$\pm\theta_1 = \left(\frac{1 - bx - (1 - 2bx + b^2x^2 - 4acx^2)^{\frac{1}{2}}}{2a}\right)^{\frac{1}{2}}$$

and

$$\pm\theta_2 = \left(\frac{1 - bx + (1 - 2bx + b^2x^2 - 4acx^2)^{\frac{1}{2}}}{2a}\right)^{\frac{1}{2}} ,$$

and the singularities $\pm\theta_1$ tend to 0 with x while the singularities $\pm\theta_2$ do not. Thus, the contour C must include $\pm\theta_1$ and exclude $\pm\theta_2$; using the residue theorem, we easily calculate

$$\frac{1}{2\pi i}\int_C \frac{-s\,ds}{a(s - \theta_1)(s + \theta_1)(s - \theta_2)(s + \theta_2)} = \frac{1}{a(\theta_2^2 - \theta_1^2)} .$$

Substituting the values of θ_1 and θ_2 given above yields the desired result. A generalization of Problems A and B can be given as follows:

Let

$$F(x,y) = \sum_{m,n} f(m,n)\, x^m y^n = (1 - ax^k - bxy - cy^k)^{-1}$$

and let

$$F(x) = \sum_{n} f(n,n)\, x^n \quad .$$

Then according to the theorem cited above, we have

$$F(x) = \frac{1}{2\pi i} \int_C \frac{-s^{k-1}\, ds}{as^{2k} - (1 - bx)s^k + cx^k}$$

Set $\omega = e^{2\pi i/k}$, then the singularities of the integrand are $\omega^j \theta_1, \omega^j \theta_2$ for $j = 1, \ldots, k$, where

$$\theta_1 = \left(\frac{1 - bx - (1 - 2bx + b^2x^2 - 4acx^k)^{\frac{1}{2}}}{2a}\right)^{1/k} ,$$

$$\theta_2 = \left(\frac{1 - bx + (1 - 2bx + b^2x^2 - 4acx^k)^{\frac{1}{2}}}{2a}\right)^{1/k} ,$$

Since θ_1 tends to 0 with x and θ_2 does not, C includes the singularities $\omega^j \theta_1$ for $j = 1, \cdots, k$, but excludes the singularities $\omega^j \theta_2$ for $j = 1, \cdots, k$. Now we have the residue theorem to find that

$$F(x) = \frac{1}{2\pi i} \int_C \frac{-s^{k-1}\, ds}{(s^k - \theta_1^k)(s^k - \theta_2^k)a} = \frac{1}{2\pi ik} \int_C \frac{-dt}{(t - \theta_1^k)(t - \theta_2^k)a} = \frac{1}{a(\theta_2^k - \theta_1^k)}$$

$$= (1 - 2bx + b^2x^2 - 4acx^k)^{-\frac{1}{2}} ,$$

where C' is a contour in the t-plane which encircles the singularity θ_1^k, k times but excludes θ_2^k.

Also solved by D. V. Jaiswal and the Proposer.

HIDDEN IDENTITY

H-153 Proposed by J. Ramanna, Government College, Mercara, India.

Show that

$$\text{(i)}\quad 4\sum_{0}^{n} F_{3k+1}F_{3k+2}(2F_{3k+1}^2 + F_{6k+3})(2F_{3k+2}^2 + F_{6k+3}) = F_{3n+3}^6$$

$$\text{(ii)}\quad 16\sum_{0}^{n} F_{3k+1}F_{3k+2}F_{6k+3}(2F_{6k+3}^2 - F_{3k}^2F_{3k+3}^2) = F_{3n+3}^3 .$$

Hence, generalize (i) and (ii) for F_{3n+3}^{2r}.

Solution by the Proposer.

We note that (i) and (ii) are easily verified for $k = 0$ and $k = 1$ and assume the results for $k = r$ and prove them for $k = r + 1$. Thus we need show, on subtracting (i) and (ii) for $n = r$ from (i) and (ii) for $n = r + 1$, respectively, that

$$\text{(2)}\quad \begin{aligned} &\text{(i)}\quad 4F_{3(r+1)+1}F_{3(r+1)+2}(2F_{3(r+1)+1}^2 + F_{6(r+1)+3})(2F_{3(r+1)+2}^2 + F_{6(r+1)+3}) \\ &\qquad = F_{3(r+1)+3}^6 - F_{3r+3}^6 \\ &\text{(ii)}\quad 16F_{3(r+1)+1}F_{3(r+1)+2}F_{6(r+1)+3}(2F_{6(r+1)+3}^2 - F_{3(r+1)}^2F_{3(r+1)+3}^2) \\ &\qquad = F_{3(r+1)+3}^8 - F_{3r+3}^8 . \end{aligned}$$

Equations (1) are true since

$$\text{(3)}\quad \begin{aligned} &\text{(i)}\quad 4F_{3(r+1)+1}F_{3(r+1)+2}(2F_{3(r+1)+1}^2 + F_{6(r+1)+3})(2F_{3(r+1)+2}^2 + F_{6(r+1)+3}) \\ &\qquad = (F_{3(r+1)+2} + F_{3(r+1)+1})^6 - (F_{3(r+1)+2} - F_{3(r+1)+1}) \end{aligned}$$

(ii) $$16\, F_{3(r+1)+1}F_{3(r+1)+2}F_{6(r+1)+3}(F^4_{3(r+1)+2} + 6F^2_{3(r+1)+2}F^2_{3(r+1)+1} + F^4_{3(r+1)+1})$$

$$= F^8_{3(r+1)+3} - F^8_{3r+3} = 16F_{3(r+1)+1}F_{3(r+1)+2}F_{6(r+1)+3}(2F^2_{6(r+1)+3} - F^2_{3(r+1)}F^2_{3(r+1)+3})$$

Hence, the desired results follow.

Also solved by D. V. Jaiswal and C. C. Yalavigi.

TRIPLE THREAT

H-154 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that for m,n,p integers ≥ 0,

$$\sum_{i,j,k\geq 0}\binom{m+1}{j+k+1}\binom{n+1}{i+k+1}\binom{p+1}{i+j+1}$$

$$= \sum_{a=0}^{m}\sum_{b=0}^{n}\sum_{c=0}^{p}\binom{m-a+b}{b}\binom{a-b+c}{c}\binom{p-c+a}{a} ,$$

and generalize.

Solution by the Proposer.

Put

$$S_{m,n,p} = \sum_{i,j,k\geq 0}\binom{m+1}{j+k+1}\binom{n+1}{i+k+1}\binom{p+1}{i+j+1} ,$$

$$T_{m,n,p} = \sum_{a=0}^{m}\sum_{b=0}^{n}\sum_{c=0}^{p}\binom{m-a+b}{b}\binom{n-b+c}{c}\binom{p-c+a}{a} .$$

Then

$$\sum_{m,n,p=0}^{\infty} S_{m,n,p}\, x^m y^n z^p$$

$$= \sum_{i,j,k=0} x^{j+k} y^{i+k} z^{i+j} \sum_{m=0}^{\infty} \binom{m+j+k+1}{j+k+1} x^m \sum_{n=0}^{\infty} \binom{n+i+k+1}{i+k+1} y^n$$

$$\cdot \sum_{p=0}^{\infty} \binom{p+i+j+1}{i+j+1} z^p$$

$$= \sum_{i,j,k=0}^{\infty} x^{j+k} y^{i+k} z^{i+j} (1-x)^{-j-k-2}(1-y)^{-i-k-2}(1-z)^{-i-j-2}$$

$$= (1-x)^{-2}(1-x)^{-2}(1-z)^{-2}\left(1 - \frac{yz}{(1-y)(1-z)}\right)^{-1}\left(1 - \frac{xz}{(1-x)(1-z)}\right)^{-1}$$

$$\cdot \left(1 - \frac{xy}{(1-x)(1-x)}\right)^{-1}$$

$$= (1-y-z)^{-1}(1-x-z)^{-1}(1-x-y)^{-1}.$$

In the next place,

$$\sum_{m,n,p=0}^{\infty} T_{m,n,p}\, x^m y^n z^p$$

$$= \sum_{a,b,c=0}^{\infty} x^a y^b z^c \sum_{m=0}^{\infty} \binom{m+b}{b} x^m \sum_{n=0}^{\infty} \binom{n+c}{c} y^n \sum_{p=0}^{\infty} \binom{p+a}{a} z^p$$

$$= \sum_{a,b,c=0}^{\infty} x^a y^b z^c (1-x)^{-b-1}(1-y)^{-c-1}(1-z)^{-a-1}$$

$$= (1-x)^{-1}(1-y)^{-1}(1-z)^{-1}\left(1 - \frac{x}{1-z}\right)^{-1}\left(1 - \frac{y}{1-x}\right)^{-1}\left(1 - \frac{z}{1-y}\right)^{-1}$$

$$= (1-x-z)^{-1}(1-x-y)^{-1}(1-y-z)^{-1},$$

and the result follows at once.

GENERALIZED VERSION

Let $k \geq 2$ and $n_1, n_2, \cdots, n_k$ non-negative integers. Show that

$$\sum \binom{n_1 + 1}{a_k + a_1 + 1}\binom{n_2 + 1}{a_1 + a_2 + 1} \cdots \binom{n_k + 1}{a_{k-1} + a_k + 1}$$

$$= \sum \binom{a_1 + a_2}{a_1}\binom{a_3 + a_4}{a_3} \cdots \binom{a_{2k-1} + a_{2k}}{a_{2k-1}}$$

where the first summation is over all non-negative $a_1, \cdots, a_k$ while in the second sum

$$a_{2k} + a_1 = n_1, \quad a_2 + a_3 = n_2, \quad a_4 + a_5 = n_3, \cdots, a_{2k-2} + a_{2k-1} = n_k .$$

Solution. Let $S(n_1, \cdots, n_k)$ denote the first sum and $T(n_1, \cdots, n_k)$ denote the second sum. Consider the expansion of

$$\phi = \phi(x_1, \cdots, x_k) = (1 - x_1 - x_2)^{-1}(1 - x_2 - x_3)^{-1} \cdots (1 - x_k - x_1)^{-1} .$$

Since

$$(1 - x - y)^{-1} = ((1 - x)(1 - y) - xy)^{-1} = \sum_{a=0}^{\infty} \frac{x^a}{(1 - x)^{a+1}(1 - y)^{b+1}} ,$$

we have

$$\phi = \sum_{a_1, \cdots, a_k = 0}^{\infty} \frac{x_1^{a_k + a_1} x_2^{a_1 + a_2} \cdots x_k^{a_{k-1} + a_k}}{(1 - x_1)^{a_k + a_1 + 2}(1 - x_2)^{a_1 + a_2 + 2} \cdots (1 - x_k)^{a_{k-1} + a_k + 2}}$$

$$\phi = \sum_{a_1,\cdots,a_k=0}^{\infty} x_1^{a_k+a_1} x_2^{a_1+a_2} \cdots x_k^{a_{k-1}+a_k}$$

$$\cdot \sum_{b_1,\cdots,b_k=0}^{\infty} \binom{a_k + a_1 + b_1 + 1}{a_k + a_1 + 1}\binom{a_1 + a_2 + b_2 + 1}{a_1 + a_2 + 1} \cdots$$

$$\cdots \binom{a_{k-1} + a_k + b_k + 1}{a_{k-1} + a_k + 1} x_1^{b_1} x_2^{b_2} \cdots x_k^{b_k}$$

$$= \sum_{n_1,\cdots,n_k=0}^{\infty} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} S(n_1, n_2, \cdots, n_k) .$$

On the other hand, since

$$(1 - x - y)^{-1} = \sum_{a,b=0}^{\infty} \binom{a + b}{a} x^a y^b ,$$

$$\phi = \sum_{a_1,a_2=0}^{\infty} \binom{a_1 + a_2}{a_1} x_1^{a_1} x_2^{a_2} \sum_{a_3,a_4=0}^{\infty} \binom{a_3 + a_4}{a_3} x_2^{a_3} x_3^{a_4} \cdots$$

$$\sum_{a_{2k-1},a_{2k}=0} \binom{a_{2k-1} + a_{2k}}{a_{2k-1}} x_k^{a_{2k-1}} x_1^{a_{2k}}$$

$$= \sum_{n_1,\cdots,n_k=0}^{\infty} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} T(n_1,n_2,\cdots,n_k) .$$

The stated result now follows at once.

RECURRING THEME

H-155 Proposed by M. N. S. Swamy, Nova Scotia Technical College, Halifax, Canada.

The Fibonacci polynomials are defined by

$$f_{n+1}(x) = xf_n(x) + f_{n-1}(x)$$

with $f_1(x) = 1$ and $f_2(x) = x$. Let $z_{r,s} = f_r(x)f_s(y)$. If $z_{r,s}$ satisfies the relation

$$z_{r+4,s+4} + a\,z_{r+3,s+3} + b\,z_{r+2,s+2} + c\,z_{r+1,s+1} + d\,z_{r,s} = 0 ,$$

show that

$$a = c = -xy, \quad b = -(x^2 + y^2 + 2) \quad \text{and} \quad d = 1 .$$

Solution by the Proposer.

Let $u_r = f_r(x)$ and $v_r = f_r(y)$. Then,

$$\begin{aligned} z_{r+4,s+4} &= u_{r+4}v_{s+4} = (xu_{r+3} + u_{r+2})(yv_{s+3} + v_{s+2}) \\ &= xyz_{r+3,s+3} + z_{r+2,s+2} + (xu_{r+3}v_{s+2} + yu_{r+2}v_{s+3}) \end{aligned}$$

Now,

$$\begin{aligned} &(xu_{r+3}v_{s+2} + yu_{r+2}v_{s+3}) \\ &\quad = x(xu_{r+2} + u_{r+1})v_{s+2} + y(yv_{s+2} + v_{s+1})u_{r+2} \\ &\quad = (x^2 + y^2)z_{r+2,s+2} + (xu_{r+1}v_{s+2} + yv_{s+1}u_{r+2}) \\ &\quad = (x^2 + y^2)z_{r+2,s+2} + xu_{r+1}(yv_{s+1} + v_s) + y\,v_{s+1}\,u_{r+2} \\ &\quad = (x^2 + y^2)z_{r+2,s+2} + xyz_{r+1,s+1} + xu_{r+1}v_s + u_{r+2}(v_{s+2} - v_s) \\ &\quad = (x^2 + y^2)z_{r+2,s+2} + xyz_{r+1,s+1} + v_s(xu_{r+1} - u_{r+2}) + z_{r+2,s+2} \\ &\quad = (x^2 + y^2 + 1)z_{r+2,s+2} + xyz_{r+1,s+1} + v_s(-u_r) = \end{aligned}$$

$$= (x^2 + y^2 + 1)z_{r+2,s+2} + x\,y\,z_{r+1,s+1} + z_{r,s} \;.$$

Hence,

$$z_{r+4,s+4} = x\,y\,z_{r+3,s+3} + (x^2 + y^2 + 2)z_{r+2,s+2} + x\,y\,z_{r+1,s+1} - z_{r,s} \;.$$

Thus,

$$a = -xy, \quad b = -(x^2 + y^2 + 2),$$
$$c = -xy, \quad d = 1 \;.$$

Also solved by W. Brady, D. Zeitlin, and D. V. Jaiswal.

Late Acknowledgement: D. V. Jaiswal solved H-126, H-127, H-129, H-131.

LETTER TO THE EDITOR

DAVID G. BEVERAGE
San Diego State College, San Diego, California

In regard to the two articles, "A Shorter Proof," by Irving Adler (December, 1969, Fibonacci Quarterly), and "1967 as the Sum of Three Squares," by Brother Alfred Brousseau (April, 1967, Fibonacci Quarterly), the general result is as follows:

$x^2 + y^2 + z^2 = n$ is solvable if and only if n is not of the form $4^t(8k + 7)$, for $t = 0, 1, 2, \cdots$, $k = 0, 1, 2, \cdots$.*

Since $1967 = 8(245) + 7$, $1967 \neq x^2 + y^2 + z^2$. A lesser result known to Fermat and proven by Descartes is that no integer $8k + 7$ is the sum of three rational squares.**

*William H. Leveque, Topics in Number Theory, Vol. 1, p. 133.

**Leonard E. Dickson, History of the Theory of Numbers, Vol. II, Chap. VII, p. 259.