SOME REMARKS ON THE ORDERING OF GENERAL FIBONACCI SEQUENCES

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In The Fibonacci Quarterly, Vol. 1, No. 4, December, 1963, Brother U. Alfred suggested a method whereby the general Fibonacci sequences could be ordered. The remarks of this paper are intended to supplement, rather than supplant, those of Brother Alfred. Another system of ordering is proposed herein.

We shall obtain a general Fibonacci sequence by taking any two integers, a and b, and employing the relationship $a + b = c$. Utilizing the set of integers for indexing, with $a = T_n$ and $b = T_{n+1}$, and requiring that $T_n < T_{n+1}$, we may define the general sequence by using the recursive form:

$$T_n + T_{n+1} = T_{n+2} .$$

To eliminate possible confusion, we adjust the indexing such that T_0 is the smallest non-negative term of the sequence.

It has been shown [1], and is easily verified, that when T_0 is the smallest non-negative term of any general Fibonacci sequence,

$$2T_0 < T_1 ; \quad 2T_{n+1} > T_{n+2}, \quad (n = 0, 1, 2, 3, \cdots) .$$

Thus, while any two successive terms of a general Fibonacci sequence are sufficient to define the entire sequence, we should like to employ T_0 and T_1. Hence, we have a unique representation of each general Fibonacci sequence, i.e.,

$$(T_0, T_1) = T_0, T_1, T_2, T_3, T_4, \cdots .$$

Attention is called to the fact that we do not require any two successive terms to be relatively prime and employ the single restriction that T_0 be the smallest non-negative term of the sequence.

Since we now have a unique representation of each general Fibonacci sequence, one of the next logical steps would be to devise a method of ordering the sequences. Such a method should be easy to apply, virtually by visual inspection. Given any representation of a general Fibonacci sequence, we immediately have the first two terms, the third term is a simple summation, and we can easily calculate the characteristic number, D, from the well-known relation [2]

$$T_1^2 - T_1T_0 - T_0^2 = D\,.$$

From these few properties we should like to accomplish the desired ordering of all general Fibonacci Sequences.

Utilizing the property that each sequence has a unique characteristic number, D, Brother U. Alfred has suggested a method of arranging the general Fibonacci sequences with respect to the value of D. Where several sequences have the same D, the size of T_0 becomes the second criterion. With the restriction that successive terms of the sequence be relatively prime, no doubt to eliminate multiples of a sequence, he suggests the following convention:

Given:	D	(T_0, T_1)
	1	(0,1)
	5	(1,3)
	11	(1,4), (2,5)
	19	(1,5), (3,7)
	29	(1,6), (4,9)
	31	(2,7), (3,8)
	41	(1,7), (5,11)
	55	(1,8), (6,13)
	⋮	

Let S_n', $(n = 1, 2, 3, 4, \cdots)$, denote the n^{th} sequence of the ordering, and we have

$$S_1' = (0,1),\quad S_2' = (1,3),\quad S_3' = (1,4),\quad S_4' = (2,5),\quad S_5' = (1,5),$$
$$S_6' = (3,7),\quad S_7' = (1,6),\quad S_8' = (4,9),\quad S_9' = (2,7),\quad S_{10}' = (3,8),\ \cdots\,.$$

By dropping the restriction that any two successive terms be relatively prime, but retaining the same principle, we must then modify the above method as follows:

Given:

D	(T_0, T_1)
1	(0,1)
4	(0,2)
5	(1,3)
9	(0,3)
11	(1,4), (2,5)
16	(0,4)
19	(1,5), (3,7)
20	(2,6)
25	(0,5)
29	(1,6), (4,9)
31	(2,7), (3,8)
36	(0,6)
41	(1,7), (5,11)
44	(2,8), (4,10)
45	(3,9)
⋮	

Hence, we now have, with S''_n $(n = 1, 2, 3, 4, \cdots)$ denoting the n^{th} sequence of the modified ordering

$$S''_1 = (0,1),\quad S''_2 = (0,2),\quad S''_3 = (1,3),\quad S''_4 = (0,3),\quad S''_5 = (1,4),$$
$$S''_6 = (2,5),\quad S''_7 = (0,4),\quad S''_8 = (1,5),\quad S''_9 = (3,7),\quad S''_{10} = (2,6), \cdots .$$

Let us examine three of the sequences, observing the above systems of ordering.

$$S'_8 = S''_{13} = (4,9) = 4, 9, 13, 22, 35, 57, 92, \cdots \quad D = 29$$
$$S'_{11} = S''_{17} = (1,7) = 1, 7, 8, 15, 23, 38, 61, \cdots \quad D = 41$$
$$S'_{12} = S''_{18} = (5,11) = 5, 11, 16, 27, 43, 69, 112, \cdots \quad D = 41 .$$

Here note that there is no simple relationship between the k^{th} term of S'_1 and S'_j (or S''_i and S''_j) as compared to the relationship of i and j, thus eliminating one of the most desirable results we should like to have from a system of ordering. For example: Given S'_{23} and S'_{25}, is the k^{th} term of S'_{23} smaller than the k^{th} term of S'_{25}?

Furthermore, additional investigation has not revealed any simple relation for comparing k^{th} terms of individual sequences using the readily available information we should like to use for an ordering. Thus it appears, at least for the present, we shall have to be content with a method of arranging the general Fibonacci sequences so that they may be designated, or counted, for example:

$$S'_{1070} = (?,?) \qquad \text{or} \qquad (237,475) = S'_7 \ .$$

Utilizing a table of D's up to a value of 1000, one finds, for D greater than 5, at least two sequences associated with each D. Occasionally, but without a recognizable pattern,* four sequences are found to be associated with a particular D. Therefore, without extensive calculation or the aid of a lengthy table, we would not know how many sequences precede a sequence having a D in, for example, the 1,000,000 region. Hence, an ordering, using D as an index, becomes unwieldy with larger values of D, the characteristic number of the sequence.

With the above limitation in mind, an alternate proposal for ordering is presented herein. There is no noteworthy advantage claimed, other than the convenience of obtaining S_n or (T_0, T_1) where large indices and/or initial terms are involved. Again, there is no simple relationship between the k^{th} term of S_i as compared to the k^{th} term of S_j.

Let us arrange the unique representation of each general Fibonacci sequence, (T_0, T_1), in an infinite matrix array in the following manner:

$$\begin{array}{llllll}
(0,1) & (0,2) & (0,3) & (0,4) & (0,5) & \cdots \\
(1,3) & (1,4) & (1,5) & (1,6) & (1,7) & \cdots \\
(2,5) & (2,6) & (2,7) & (2,8) & (2,9) & \cdots \\
\vdots & & & & & \\
(j,2j+1) & (j,2j+2) & (j,2j+3) & (j,2j+4) & (j,2j+5) & \cdots \\
\vdots & & & & &
\end{array}$$

$$(j = 0, 1, 2, 3, 4, \cdots) \ .$$

*See [5] and [6].

Given the above display of general Fibonacci sequence representations, the remaining problem is that of choosing the system of ordering to be employed. Note that the representation is in the $T_0 + 1$ row and the $T_1 - 2T_0$ column. Two methods that might be considered are shown below, together with certain comments. Observe that the numbers in the following displays are those of the indexing set and only reflect the position of, or number assigned to, each general Fibonacci sequence.

I. Diagonal Method

1	2	4	7	11	16	22	⋯
3	5	8	12	17	23		
6	9	13	18	24			
10	14	19	25				
15	20	26					
21	27						
28							
⋮							

There are several methods of associating the proper row and column of any given integer in the above display of positive integers, thus each reader is free to choose his own favorite scheme. Using the diagonal method of ordering, with S_n denoting the n^{th} sequence, we obtain

$S_{1070} = (?,?) = (34,80)$ 1070 appears in the 35th row and the 12th column. $T_0 + 1 = 35$ and $T_1 - 2T_0 = 12$.

$(237,475) = S_? = S_{28411}$ 28411 is found in the 237 + 1 row and the 475 - 2 x 237 column.

Only a short calculation is required to obtain the desired information, n or S_n, or (T_0, T_1), when given the corresponding data. To retain consistency, the indexing set is the set of positive integers, $(n = 1, 2, 3, 4, \cdots)$.

II. Modified Sides of Squares Method

1	2	5	10	17	26	37	⋯
3	4	6	11	18	27	38	
7	8	9	12	19	28	39	
13	14	15	16	20	29	40	
21	22	23	24	25	30	41	
31	32	33	34	35	36	42	
43	44	45	46	47	48	49	
⋮							

Again, the reader is free to use any one of several well-known methods of obtaining the row and column of a given integer, and we observe, with S_n denoting the n^{th} sequence of the ordering $(n = 1, 2, 3, 4, \cdots)$

$S^*_{1070} = (?,?) = (32,78)$	1070 is in the 33^{rd} row and the 14^{th} column.
$(237,475) = S^*_? = S^*_{56407}$	The 238^{th} row and the 1^{st} column is the position of 56407.

With regard to the two demonstrated methods of ordering, the diagonal is possibly a more elegant attack for it parallels that used to count other infinite matrix arrays.

In conclusion, we have proposed another method of arranging or ordering the general Fibonacci sequences — the first being that as suggested by Brother U. Alfred. Undoubtedly, there are still others. As implied earlier, this property of a unique representation of each general Fibonacci sequence demands the ultimate adoption of some system of ordering to assist the growing number of Fibonacci devotees. References 3, 4, 5, and 6 were added by the Editor.

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