COMBINATORIAL PROBLEMS FOR GENERALIZED FIBONACCI NUMBERS

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Theorem 1. The number of subsets of $\{1, 2, 3, \cdots, n\}$ which have k elements and satisfy the constraint that i and $i + j$ $(j = 1, 2, 3, \cdots, a)$ do not appear in the same subset is

$$f_a(n,k) = \binom{n - ka + a}{k} ,$$

where $\binom{n}{k}$ is the binomial coefficient. We count ϕ, the empty set, as a subset.

Comments. Before proceeding with the proof, we note with Riordan [1], that for $a = 1$, the result is due to Kaplansky. If, for fixed n, one sums over all k-part subsets, he gets Fibonacci numbers,

$$F_{n+1} = \sum_{k=0}^{[(n+1)/2]} \binom{n - k + 1}{k} , \quad (n \geq 0)$$

where $[x]$ is the greatest integer function. The theorem above is a problem given in Riordan [2].

Proof. Let $g_a(n,k)$ be the number of admissible subsets selected from the set $\{1, 2, 3, \cdots, n\}$. Then

$$g_a(n + 1,k) = g_a(n,k) + g_a(n - a, k - 1) ,$$

since $g_a(n,k)$ counts all admissible subsets without element $n + 1$ while $g_a(n - a, k - 1)$ counts all the admissible subsets which contain element $n + 1$. If element $n + 1$ is in any such subset, then the elements $n, n - 1, n - 2, n - 3, \cdots, n - a + 1$ cannot be in the subset. We select $k - 1$ elements from the $n - a$ elements $1, 2, 3, \cdots, n - a$ to make admissible subsets and add $n + 1$ to each subset. The count is precisely $g_a(n - a, k - 1)$.

Consider

$$f_a(n,k) = \binom{n - ka + a}{k}, \qquad k \geq 0 .$$

But, since the $f_a(n,k)$ are binomial coefficients,

$$f_a(n + 1,k) = \binom{n + 1 - ka + a}{k} = \binom{n - ka + a}{k} + \binom{n - a - (k-1)a + a}{k - 1}$$

$$= f_a(n,k) + f_a(n - a,\ k - 1) .$$

Thus, $f_a(n,k)$ and $g_a(n,k)$ satisfy the same recurrence relation. Since the boundary conditions are

$$g_a(n,1) = f_a(n,1) = n ,$$

and

$$g_a(1,n) = g_n(1,n) = 0, \qquad n > 1 ,$$

the arrays are identical. This concludes the proof of Theorem 1.

We note that, for fixed $k \geq 0$, the number of k-part subsets of $\{1, 2, 3, \cdots, n\}$ for $n = 0, 1, 2, \cdots$, are aligned in the k^{th} column of Pascal's left-adjusted triangle. If one sums for fixed n the number of k-part subsets, one obtains

$$V_a(n,a) = \sum_{k=0}^{\left[\frac{n+a}{a+1}\right]} f_n(n,k) = \sum_{k=0}^{\left[\frac{n+a}{a+1}\right]} \binom{n - ka + a}{k} ,$$

where $[x]$ is the greatest integer function. These are precisely the generalized Fibonacci numbers of Harris and Styles [3]. There,

$$u(n;p,1) = \sum_{k=0}^{[n/(p+1)]} \binom{n - kp}{k}$$

so that

$$V_a(n,a) = u(n + a;\ n,1) \ .$$

Clearly, if we select only certain k-part subsets $(b \geq 1)$

$$V_a(n,a,b) = \sum_{k=0}^{\left[\frac{n+a}{a+b}\right]} \binom{m - ka + a}{kb}$$

then

$$V_a(n,a,b) = u(n + a;\ a,b) \ .$$

Thus, one has a nice combinatorial problem in restricted subsets whose solution sequences are the generalized Fibonacci numbers defined in [3] and studied in [4], [5], [6], [7], [11], and [12].

GENERALIZATION

We extend Theorem 1 to all generalized Pascal triangles.

Theorem 2. The number of subsets of $\{1, 2, 3, \cdots, n\}$ with k elements in which $i,\ i + j$ $(j = 1, 2, \cdots, a)$ are not in the same subset nor are simultaneously all of the integers $i + ja + 1$ $(j = 0, 1, 2, \cdots, r - 1)$, in the same subset, is

$$f_a(n,k,r) = \left\{ \begin{matrix} n - ka + a \\ k \end{matrix} \right\}_r ,$$

where

$$(1 + x + x^2 + \cdots + x^{r-1})^n = \sum_{i=0}^{n(r-1)} \left\{ {n \atop i} \right\}_r x^i .$$

We call

$$\left\{ {n \atop i} \right\}_r$$

the r-nomial coefficients, and n designates the row and i designates the column in the generalized Pascal triangle induced by the expansion of

$$(1 + x + x^2 + \cdots + x^{r-1})^n , \quad n = 0,1,2,\cdots .$$

Proof. Let $g_a(n,k,r)$ be the number of admissible subsets selected with elements from $\{1, 2, 3, \cdots, n\}$. Then

$$g_a(n + 1,k,r) = g_a(n,k,r) + g_a(n - a,k - 1,r) + g_a(n - 2a,k - 2,r) + \cdots + g_a(n - (r - 1)a, k - r + 1,r)$$

Consider the set of numbers $n + 1$, $n - a + 1$, $n - 2a + 1$, $n - 3a + 1$, $\cdots$, $n - (r - 1)a + 1$. The general term $g_a(n - sa, k - s,r)$ gives the number of admissible subsets which require the use of $n + 1$, $n - a + 1$, $n - 2a + 1$, $\cdots$, $n - (s - 1)a + 1$, disallows the integer $n - sa + 1$, but permits the use of the integers $n = 1, 2, 3, \cdots, n - sa$ in the subsets subject to the constraints that integers i, $i + j$ $(j = 1, 2, 3, \cdots, a)$ do not appear in the same subset. This concludes the derivation of the recurrence relation.

Next, consider

$$f_a(n,k,r) = \left\{ {n - ka + a \atop k} \right\}_r$$

Since $f_a(n,k,r)$ is an r-nomial coefficient, then

$$f_a(n + 1,k,r) = f_a(n,k,r) + f_a(n - a, k - 1,r) + \cdots + f_a(n - sa, k - s,r) + \cdots + f_a(n - (r - 1)a, k - r + 1,r) .$$

Thus, $f_a(n,k,r)$ and $g_a(n,k,r)$ both obey the same recurrence relation, and

$$f_a(n,1,r) = g_a(n,1,r) = n$$
$$f_a(1,n,r) = g_a(1,n,r) = 0, \quad n > 1$$

for all $n \geq 0$, so that the arrays are identical for all $k \geq 0$.

Summing, for fixed $n \geq 0$, over all numbers of all k-part subsets yields

$$V_a(n,a,r) = \sum_{k=0}^{\left[\frac{(n+a)(r-1)}{1+a(r-1)}\right]} \left\{ \begin{matrix} n - ka + a \\ k \end{matrix} \right\}_r$$

If we now generalize the "generalized Fibonacci numbers, $u(n; p,q)$, of Harris and Styles [3]" to the generalized Pascal triangles obtained from the expansions $(1 + x + x^2 + \cdots + x^{r-1})^n$, $n = 0, 1, 2, 3, \cdots$,

$$u(n; p,q,r) = \sum_{k=0}^{\left[\frac{n(r-1)}{q+p(r-1)}\right]} \left\{ \begin{matrix} n - kp \\ kq \end{matrix} \right\}_r ,$$

there are precisely

$$p + \left[\frac{q}{r-1}\right] + 1$$

ones at the beginning of each $u(n; p,q,r)$ sequence. Our application starts with just one 1. Let

$$m = \left[\frac{q}{r-1}\right] ,$$

the greatest integer in $q/(r-1)$. Then,

$$u(n + a + m; a,b,r) = \sum_{k=0}^{\left[\frac{(n+a+m)(r-1)}{b+a(r-1)}\right]} \left\{ \begin{matrix} n + a + m - ka \\ kb \end{matrix} \right\}_r$$

Thus the solution set to the number of subsets of $\{1, 2, 3, \cdots, n\}$ subject to the constraints that no pairs i, $i + j$ $(j = 1, 2, 3, \cdots, a)$ are to be allowed in the same subset, nor are all of $i + ja + 1$ $(j = 0, 1, 2, 3, \cdots, r - 1)$ to be allowed in the same subset, are the generalized Fibonacci numbers of Harris and Styles generalized to Pascal triangles induced from the expansions of

$$(1 + x + x^2 + \cdots + x^{r-1})^n, \qquad n = 0, 1, 2, 3, \cdots .$$

One notes that the r-nacci generalized Fibonacci numbers

$$u(n; 1,1,r) = \sum_{k=0}^{\left[\frac{n(r-1)}{r}\right]} \left\{ \begin{matrix} n - k \\ k \end{matrix} \right\}_r$$

are not generally obtained by setting $a = 0$ in the above formulation. However, the generalized Fibonacci sequences for the binomial triangle are obtained if $r = 2$. The other r-nacci number sequences are obtained if the subsets are simply restricted from containing simultaneously r consecutive integers from the set $\{1, 2, 3, \cdots, n\}$ but there is no restriction of $r > 2$ about pairs of consecutive integers. Thus, for these r-nacci sequences $(r > 2)$, we cannot simply set $a = 1$. However, the formulas look identical.

Let

$$V(n; 1,1,r) = u(n + 1; 1,1,r) ;$$

then

$$V(n;\, 1, 1, r) = \sum_{k=0}^{\left[\frac{(n+1)(r-1)}{r}\right]} \left\{ {n - k + 1 \atop k} \right\}_r$$

which is seen to be the generalization of Kaplansky's lemma to generalized Pascal triangles.

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