THE EDUCATIONAL VALUE IN MATHEMATICS

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The educational value in mathematics is higher to the perceptive individual than the manufacture of things and the mere solving of problems.

I would like to offer a different approach to the significance of the Golden Ratio. It is for the purpose of making clear to the layman that beyond the useful pursuit of mathematics for its own sake (which is not an end in itself) lies the deep philosophic content, and a content that constitutes a very important ingredient of philosophy.

Plato's Divided Line and Euclid's Golden Section bear identical ratios. From Plato's exposition, we derive much of the philosophic content.

Let me begin by noting that the Creative Right Triangle of Pythagoras (3, 4, 5) is itself created by the encompassing environment of three Right Triangles whose perpendiculars bear the relation of 2 to 1. This type of right triangle will be termed, "the Celestial Right Triangle," occupying three-tenths of a Square. See Figure 1.

From this locating a cosmic position for the creative triangle, it becomes immediately inviting to search for properties of the celestial triangle which may lead to many analogies.

We will have more to say about this illustration, which is replete with Fibonacci and Lucas numbers, as we unfold the picture.

The location of the right angular point of the triangle illustrated in Fig-1 lies exactly in the following distances from the boundaries of the square: South, 1; East, 2; West, 3; and North, 4 (which is 5 x 5).

When the area of this square is "one," the area of the creative triangle is three-tenths, and as the cardinal numbers increase, the area of the creative triangle increases in the arithmetical progression of 0.3, 0.6, 0.9, etc.

Therefore, the area of the creative triangle reaches identity with cardinal numbers at its intervals in the arithmetical progression of 10 when its area becomes 3, 6, 9, 12, etc.

When the area of the square is expressed by cardinal numbers squared, the areas of the creative triangles increase by an increasing progression of 4, 9, 16, 25, etc. The right triangle has sides 3, 4, 5 multiplied by $\sqrt{5}/2$.

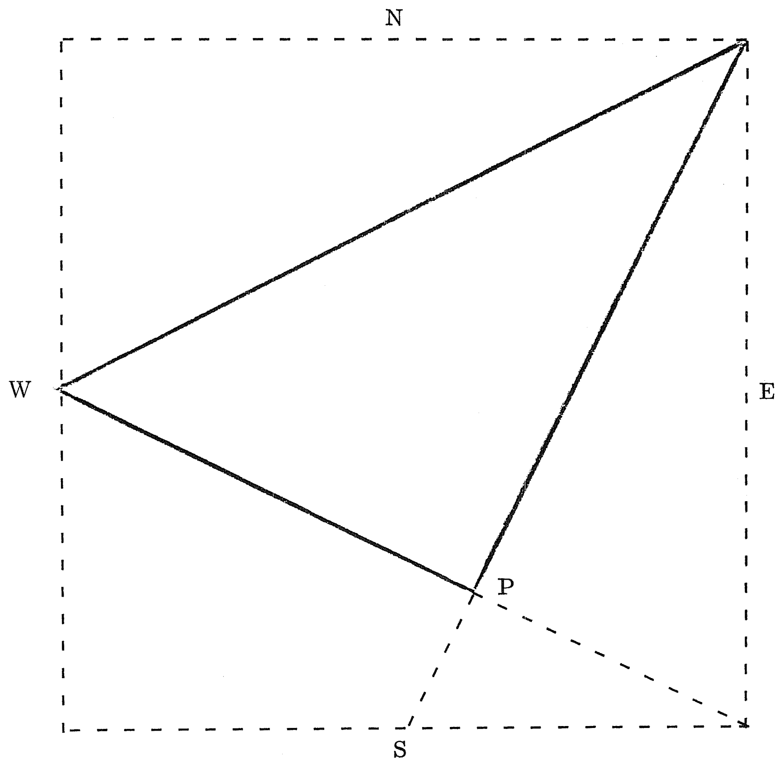


Fig. 1 The Celestial Right Triangle

When the areas of the creative triangles reach our squared numbers, the encompassing square is ten-thirds of them; that is, if the area of the creative triangle is 36, that square is 120. This indicates a series of Cosmic squares in the progression of

10,	20,	30,	53-1/3,	83-1/3,
120,	163-1/3,	213-1/3,	270,	333-1/3,
403-1/3,	480,	563-1/3,	653-1/3,	750, etc.

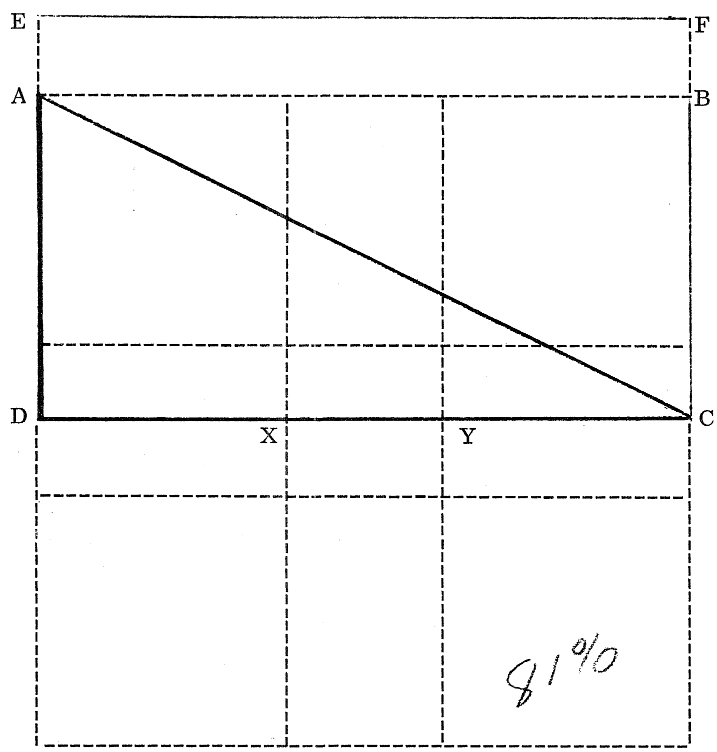
From Figure 1, this means that we have another Cosmic series of squares, formed with progressive side lengths of $\sqrt{5}$, $2\sqrt{5}$, $3\sqrt{5}$, $4\sqrt{5}$, etc., when the square area becomes

5, 20, 45, 80, 125, 180, 245, 320, 405, 500, 605, 720, etc.

The areas of the creative triangle are three-tenths of them, or, in the series of 6, 24, 54, 96, 150, 216, etc., and therefore in the increasing progression of 4, 9, 16, 25, etc.

In Figure 2, the point of Golden Section is determined at once by taking the difference between the length of the hypotenuse and the length of the greater perpendicular, and adding this difference to the length of one-half of this perpendicular.

Dr. Verner E. Hoggatt, Jr., in his book, Fibonacci and Lucas Numbers, ably explains the same situation by a different method of construction. The square thus divided into these ratio segments, yields in three dimensions, 27 prisms to form the cube. Among them, only four have variant volumes.



Line Lengths

$$DC = 1$$

$$DA = 0.5$$

$$AC = \sqrt{5}/2$$

$$^{*}AE = \frac{\sqrt{5}}{2} - 1 = \frac{\sqrt{5} - 2}{2} = \frac{1}{2\sqrt{5} + 4}$$

$$AE = \frac{1}{2\sqrt{5} + 4} = \frac{XY}{2}$$

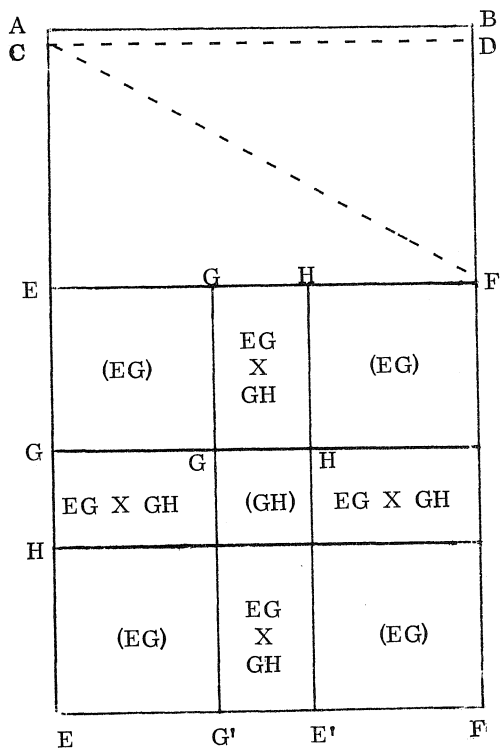
$$^{**}DE = \frac{1}{2} + \frac{1}{2\sqrt{5} + 4} = \frac{2}{1 + \sqrt{5}} = \alpha^{-1}$$

$$DX = YC = \frac{2}{\sqrt{5} + 3} = \alpha^{-2} \qquad DY = XC = \frac{2}{\sqrt{5} + 1} = \alpha^{-1} \qquad XY = \frac{1}{\sqrt{5} + 2} = \alpha^{-3}$$

*Because the Hypotenuse AC minus the greater perpendicular DC equals XC.
**DE then becomes the length of the Golden Section, equal to DY and XC.

Fig. 2 The point of Golden Section located by Simple Subtraction and Addition

In Figure 3, Plato had a term for the lines EH and GF, which he called, "the Intelligible."



Line Lengths

$$AC = \frac{1}{2\sqrt{5}+4}$$

$$GH = \frac{1}{\sqrt{5}+2}$$

$AB = 1 \qquad EC = 0.5$

$$CF = \frac{\sqrt{5}}{2}$$

$$EG = \frac{2}{\sqrt{5}+3}$$

$$EH = \frac{2}{\sqrt{5}+1}$$ The Golden Ratio

Areas of Right Triangles CEF and CDF are each 0.25.

Areas of Golden Rectangles

$(EG)^2 = \frac{2}{3\sqrt{5}+7}$ $\quad (EH)^2 = \frac{2}{\sqrt{5}+3}$ $\quad ABFE = \frac{2}{\sqrt{5}+1}$ $\quad GHHG = \frac{2}{5\sqrt{5}+11}$

$GEG'G = \frac{2}{3\sqrt{5}+7}$ $\quad GHH'G' = \frac{1}{\sqrt{5}+2}$ $\quad EEGG' = \frac{2}{\sqrt{5}+3}$ $\quad EEHH' = \frac{2}{\sqrt{5}+1}$

Cube of Intelligence: $\quad (EH)^3 = \frac{1}{\sqrt{5}+1}$

Cube of Mathematics

$(GH)^3 = \frac{1}{17\sqrt{5}+38}$ $\quad (GH)^2 = \frac{1}{4\sqrt{5}+9}$ $\quad (EG)^3 = \frac{1}{4\sqrt{5}+9}$

Fig. 3 Illustrating the Segmented Areas and Volumes

It may be observed in Fig. 3 that the cube of the "intelligible" is composed of eight prisms, four only with variant volumes. Their volumes are:

$$(EG)^3 = \frac{1}{4\sqrt{5} + 9} \qquad \text{One}$$

$$(EG)^2 \times GH = \frac{2}{12\sqrt{5} + 29} \qquad \text{Three}$$

$$(GH)^2 \times EG = \frac{2}{21\sqrt{5} + 47} \qquad \text{Three}$$

$$(GH)^3 = \frac{1}{17\sqrt{5} + 38} \qquad \text{One}$$

Plato termed the line GH, "Mathematics."

This cube of the "intelligible" has eight positions upon the cube, with one important feature — that they all share in their construction the center cube, the cube of "mathematics."

These denominators each share numbers appearing in both the Fibonacci and Lucas series.

The sum of these eight prisms, that is, the volume of the cube of the "intelligible," is

$$\alpha^{-3} = \frac{2}{2\sqrt{5} + 4} = \frac{1}{\sqrt{5} + 2} ,$$

the difference between ratios.

Editorial Note: If

$$\alpha = \frac{1 + \sqrt{5}}{2} ,$$

then

$$\alpha^n = \frac{L_n + F_n\sqrt{5}}{2}$$

thus,

$$(EG)^3 = \frac{2}{8\sqrt{5} + 18} = \alpha^{-6}, \quad \frac{2}{13\sqrt{5} + 29} = \alpha^{-7}, \quad \frac{2}{21\sqrt{5} + 47} = \alpha^{-8}, \text{ and}$$

$$(GH)^3 = \frac{2}{34\sqrt{5} + 76} = \alpha^{-9}$$

V.E.H.

Glancing again at Fig. 1, an illustration of the "birth" of the creative triangle, you may have noted that the number 1234 is not divisible by "eleven." See Fig. 4 and accompanying text.

But, suppose we observe these distances in order of rotation. We find they run as follows: 1243, 1342, 2134, 2431, 4213, and 4312, 3124, 3421.

These quaternaries are divisible by 11, and the same holds good for the remaining possible forty. Figure 4 illustrates the potential of 48 out of 144 combinations.

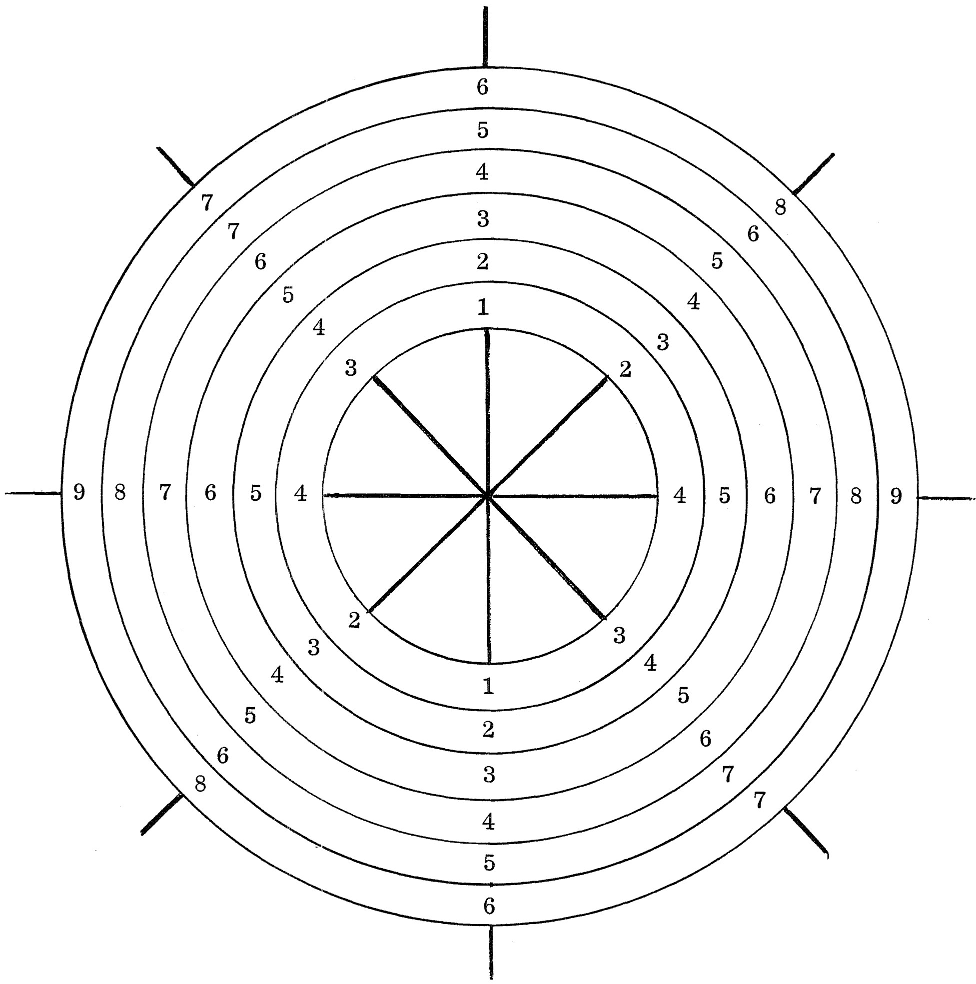
The relative proportional areas in Fig. 5 are: -1, 4, 4, 6, 5, or 1, 4, 4, 11. By rotating the triangle into its eight possible positions within the square, we obtain 24 points which coincide exactly with the points of intersection of perpendicular and horizontal lines within the square of 60 x 60.

By plotting these points, we are provided with the center of the inscribed circle, at $x = 0$, $y = 0$; and by bisecting the triangle, we have the point, $x = 0$, $y = 15$ shown in Fig. 5.

Upon making a plotted graph for each of the other triangular positions, the plotted values of the triangle are merely a matter of sign and number interchange. The inscribed circle will roll around the circumference of the circumscribed circle 5 x 0.5 times. The area of the circumscribed circle contains the area of the smaller circle 2.5 x 2.5 times. Area of triangle BDP is 90. The area of triangle EOP is 20 and 5/11ths. Multiplying all areas by 11 to clear denominator, we have a total area of xx c 1080 = 11,880. Therefore, the 3,600 square units each enjoy an area of eleven.

Such circumferences increase in the arithmetical progression of $3.6\sqrt{5} \times \pi$.

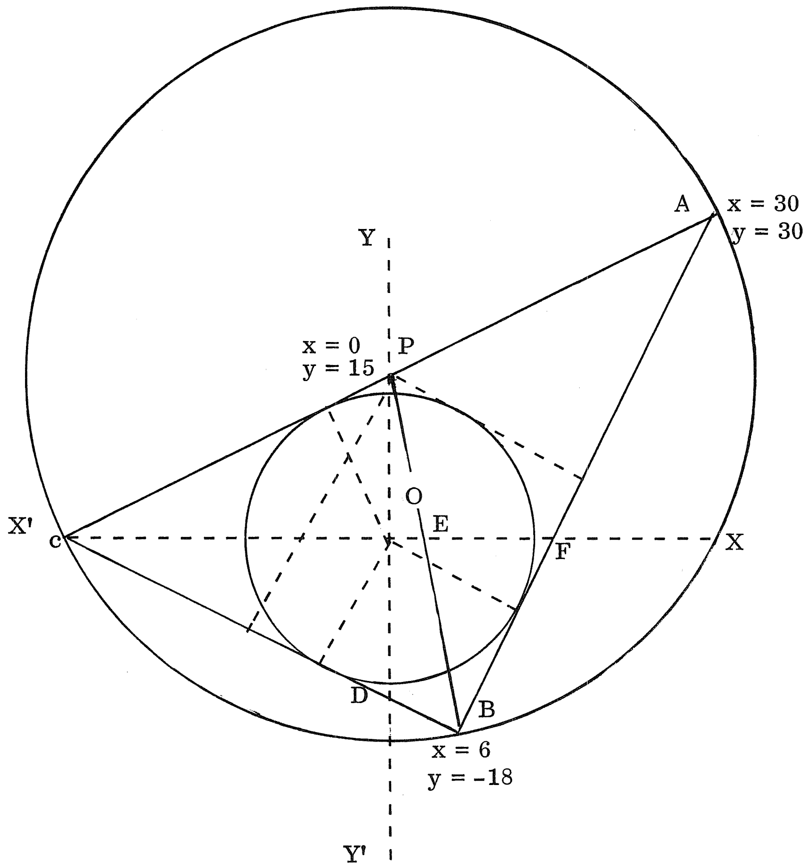
Coefficients of $\sqrt{5}$	Times Increased
11.30976	0.5
22.61952	1.0
33.92928	1.5
45.23904	2.0
56.5488	2.5
67.85856	3.0



This illustration gives the numerical evidence at a glance. Any four numbers taken in their rotary position of sequence within the same circle are divisible by eleven. These 48 numbers, out of a total of 144 possible combinations are unique in their divisibility by eleven. The remaining 96 are not exactly divisible by eleven.

This chart is not intended to bear any idea of magic, but it does reveal analogies to known Law. The causation of this law is shown in the previous pages. Philosophic research is just as rewarding as scientific research. "Ominia numeris sita sunt." (All things lie veiled in numbers.)

Fig. 4 The Number "Eleven" Chart



AREAS

Quadrilaterals		Triangles			
APEF	4725	PCO	2475	POE	225
BDCE	765	COD	2475	FBE	1215

Total of the six areas: 11,800

Fig. 5 The Number "Eleven" Chart

Coefficients of $\sqrt{5}$	
	Inscribed Circumference
37.6992	$12 \times \sqrt{5} \times \pi$
	Increase
56.5488	$18 \times \sqrt{5} \times \pi$
	Circumscribed Circumference
94.248	$30 \times \sqrt{5} \times \pi$

Factors: $4 \times 6 \times 11 \times 17 \times 21 \times 0.001$
$= 94.248$

The sums and products of these quantities yield two quantities in both numerators and denominators.

Both Fibonacci and Lucas numbers are represented in these numerators and denominators.

This indicates the home of spirals making the dynamic symmetry within the Golden Rectangles.

The Dual Quantities	In the Process of Addition And Multiplication
3 and 7	$987 = (17 \times 29) + (13 \times 38)$
	$493 + 494$
4 and 9	
5 and 11	$1220 = (21 \times 29) + (13 \times 47)$
	$609 + 611$
13 and 29	
17 and 38	$2207 = (17\sqrt{5} \times 13\sqrt{5}) + (38 \times 29)$
	$1105 + 1102$
21 and 47	
34 and 76	$2728 = (21\sqrt{5} \times 13\sqrt{5}) + (29 \times 47)$
	$1365 + 1363$
55 and 123	
89 and 199	
89 and 398	
233 and 521	
1597 and 3571	

ERRATA

Please make the following corrections in "Sums Involving Fibonacci Numbers," Vol. 7, No. 1, pp. 92-98:

The first half of Eq. (3), line 3, page 95, should read as follows:

$$\sum_{r=0}^{n} v_r(p,q) = 2 + \frac{pT_n - 2qT_{n-1}}{1 - p + q} .$$

Please make the following corrections in "Identities Involving Generalized Fibonacci Numbers," Vol. 7, No. 1, pp. 66-72:

Page 67 — Please correct line 5 to read:
It is also easy to see that $H_n = pF_n + qF_{n-1}$ where F_n is the n^{th} Fibonacci

Page 69 — Please change the last part of the last sentence of page to read:.. for the Fibonacci numbers we get in the generalized Fibonacci numbers the identity: ...

Page 71 — Please correct Eq. (26) to read as follows:

$$\sum_{r=1}^{n} H_{2r-1}^3 = \frac{1}{4}[(H_{2n}^3 - q^3) + 3e(H_{2n} - q)] .$$