THE POWERS OF THREE

J. M. WILLIAMS, JR.
San Francisco, California

Any number may be expressed in powers of three by addition or subtraction of the numbers those powers represent.

$$13 = 3^2 + 3^1 + 3^0$$
$$14 = 3^3 - 3^2 - 3^1 - 3^0 .$$

The powers used in such expressions are whole integers, no fractional powers being involved.

The number of terms required to express a number approximates twice the number of digits in the number; the greater the number of digits required the more closely this limit is approached.

Any such expression of a number need contain no repetition of any given power.

Such expressions are easily handled in arithmetic processes by observation of algebraic rules regarding exponents.

Discussion of this digital system follows.

It will be noted from Table 1 that the powers of three follow a routine sequence in the expressions for the numbers, appearing first behind the positive sign, then changing to the negative sign, and then disappearing from the statement.

The appearance of powers in the statements follows a fixed sequence:

1. The power appears for the first time at a number equal to one-half of the value of that power with 1 added.
2. It appears behind the positive sign.
3. It remains in the statement, and behind the positive sign, for a series of statements equal in number to the value of that power of three.
4. It then becomes negative in the following statement.
5. It remains in the statement, and behind the negative sign, for a series of statements equal in number to the value of that power of three.
6. It then disappears in the following statement.

Table 1

Number	expressed in	Powers of Three
1		$+3^0$
2		$+3^1-3^0$
3		$+3^1$
4		$+3^1+3^0$
5		$+3^2-3^1-3^0$
6		$+3^2-3^1$
7		$+3^2-3^1+3^0$
8		$+3^2-3^0$
9		$+3^2$
10		$+3^2+3^0$
11		$+3^2+3^1-3^0$
12		$+3^2+3^1$
13		$+3^2+3^1+3^0$
14		$+3^3-3^2-3^1-3^0$
15		$+3^3-3^2-3^1$
16		$+3^3-3^2-3^1+3^0$
17		$+3^3-3^2-3^0$
18		$+3^3-3^2$

7. It remains out of the statement for a series of statements equal in number to the value of that power of three.
8. It then reappears in the statement, and behind the positive sign, and repeats the sequence outlined above without limit.

Determination of the proper statement for any given number is outlined below.

Determination of Statement

1. Subtract 1 from the given number.
2. Divide the remainder by 3, setting the quotient below the dividend and the remainder from the division to the right, whether this remainder be 2, or 1, or 0.

3. If the remainder after division is 1 or 2 proceed as directed below, but whenever the remainder after division is 0 it is necessary to subtract 1 from the quotient before proceeding to treat it as a dividend, as below.
4. Divide again by 3 as directed in step 2, and continue this process until the dividend is 0 with 0 remainder, watching throughout the process outlined in Step 3.
5. The column of remainders 2, or 1, or 0, which have been set to the right is now numbered, beginning at the top with 0 and proceeding with 1, 2, 3, etc., and ending with the highest number in the sequence opposite the final 0 remainder. The numbers in this sequence are the powers of three.
6. Fixation of signs for the various powers, or exclusion from the statement, is determined by the remainders:
   When the remainder is 0 the sign is positive.
   When the remainder is 1 the sign is negative.
   When the remainder is 2 the power is excluded from the statement.

Demonstration of this process follows:

Given number	6056			
	- 1			
Divide by 3	6055			Power
	2018	with remainder	1	0
	672	" "	2	1
	224	" "	0	2
	- 1			
	223			
	74	" "	1	3
	24	" "	2	4
	8	" "	0	5
	- 1			
	7			
	2	" "	1	6
	0	" "	2	7
	0	" "	0	8

Consider the sequence thus, bottom to top:

Remainders:	0	2	1	0	2	1	0	2	1
Signs:	+		-	+		-	+		-
Powers:	8		6	5		3	2		0

Statement for the given number:

$$6056 = 3^8 - 3^6 + 3^5 - 3^3 + 3^2 - 3^0$$

Proof of statement:

Positive powers:	6561 + 243 + 9 =	6813
Negative powers:	- 729 - 27 - 1 =	-757
Given number:		6056

For simplicity, in further discussion, the digit 3 will not be used in power statements except when the exponent 3 is required. Statements will use only the digits designating the powers. Treated thus the statement for 6056 would be written thus: 6056 equivalent, +8 -6 +5 -3 +2 -0.

ADDITION OF STATEMENTS

In handling two or more such statements arithmetically, it is almost inevitable that there will be duplication of one or more of the powers. Consider the addition of the statements for the numbers two and three:

2 equivalent, +1 -0
3 equivalent, +1 .

Addition of +1 to the statement gives +1 +1 -0. This is re-written thus: +2 -1 -0. The next higher power is given the sign of the duplicated power, following it by the duplicated power with reversed sign; obviously 3 + 3 = 9 - 3.

When a power is triplicated, the next higher power is given the sign of the triplicated power, which is then dropped from the statement. Consider addition of the following statements:

2	equivalent,	+1 -0
3	"	+1
4	"	+1 +0
9		+2

Note that the unlike signs calcelled and removed 0 power from the statement for the sum.

When the number of repetitions exceeds three, they can be eliminated step-wise by application of the processes outlined above.

SUBTRACTION OF STATEMENTS

Subtraction is performed by changing the signs on all of the powers in the statement being subtracted and then performing addition as abovel.

MULTIPLICATION OF STATEMENTS

In performing multiplication, the digits representing the powers are added irrespective of sign, and the signs follow algebraic rule:

Multiplication of like signs yields the positive sign.

Multiplication of unlike signs yields the negative sign.

Consider multiplication of statements for 13 and 14:

14:			+3	-2	-1	-0
13				+2	+1	+0
			+3	-2	-1	-0
		+4	-3	-2	-1	
	+5	-4	-3	-2		
	+5	-4	+3	-2	+1	-0

Consider the totalling of those products step-wise:

1. The -0 comes down unchanged.
2. The duplicated -1 becomes -2 +1
3. The triplicated -2 becomes -3, with the -2 carried forward from Step 2 brought down unchanged.
4. The duplicated -3 has been triplicated by the -3 carried forward from Step 3 so it becomes -4 and the +3 is brought down unchanged.

5. The unlike signs for 4 cancel so the -4 carried forward from Step 4 is brought down unchanged.
6. The +5 is brought down unchanged.

Proof of statement:

Positive powers:	243	+ 27	+ 3	=	273
Negative powers:	-81	-9	-1	=	-91
		13 x 14		=	182

DIVISION OF STATEMENTS

This process brings another rule into play. Before stating the rule, consider these two terms, +4 -3 and +3, the statements for 54 and 27. If these terms are added, the cancellation of unlike signs will leave +4, the statement for 81. If they are subtracted, the sign representing 27 would be changed. It would then be duplicated. After treatment for duplication, it would cancel the +4 and leave +3, the statement for 27.

The rule in dividing power statements once more involves like and unlike signs, but with an extra specification added: like or unlike signs of adjacent powers as in the case of the statement for 54. The rule: when adjacent powers in the dividend have unlike signs, the two powers must be considered as a duplication of the lower power, both with the sign of the higher power. This is a reversal of the rule for the additive process and is not unreasonable since division is a subtractive process.

Demonstration of division will be the reverse of the example given for multiplication. The statement for 182 is divided by the statement for below. The notes referenced are found on the following page.

		+2	+1	+1	-0		
+2 +1 +0 /	+5	-4	+3	-2	+1	-0	See Note 1
Rewritten	+4	+4	+3	-2	+1	-0	See Note 2
	-4		-3	-2			See Note 3
		+4	-3	+2	+1	-0	See Note 4
Rewritten		+3	+3	+2	+1	-0	See Note 5
		-3		-2	-1		See Note 6
			+3			-0	See Note 7
			-3	-2	-1		See Note 8
				-2	-1	-0	See Note 9
				+2	+1	+0	See Note 10

Notes:

1. In this case, all the powers are both adjacent and unlike. The rule will be applied only to the leading pair.
2. Rewritten giving the duplicated lower power the sign of the higher power.
3. The +2 in the divisor, when subtracted from the +4 in the dividend, yields +2 for the quotient. When the sign is changed for subtraction, cancellation will result.
4. The duplicated -2 became -3 +2 and the +2 came down. Unlike signs for 3 cancelled and the -3 brought forward came down. The unaffected +4 came down, making this another case of unlike signs for adjacent powers.
5. Rewritten as in Note 2.
6. The +2 in the divisor, when subtracted from the +3 in the remainder, yields +1 for the quotient. When the sign is changed for subtraction, cancellation will result.
7. Unlike signs cancel and +3 comes down with -0.
8. The +2 in the divisor, when subtracted from the +3 in the remainder, yields +1 for the quotient. When the sign is changed for subtraction, cancellation will result. The duplication of +1 in the quotient can be cared for after completion of the division.
9. Unlike signs cancel, and unaffected terms come down.
10. The +2 in the divisor now encounters -2 in the remainder. Disregarding signs, the 2 from 2 yields 0 for the quotient. Adding -0 to +2 will yield -2; when the sign is changed for subtraction, cancellation will result.

The quotient reads +2 +1 +1 =0. There is duplication of +1. This is changed to +2 -1. This change duplicates +2. This is changed to +3 -2. Now the corrected statement reads +3 -2 -1 -0, the power statement for 14.

This is as far as I have investigated this curiosity with any success. Perhaps someone else can find a way to extract roots and raise to higher powers without simple multiplication. Decimal fractions can, of course, be handled by appropriate multiplication by powers of 10, as can uneven divisions that result in significant remainders.

I apologize for using the expression "negative powers" when values involved are not reciprocals. This misuse was a short-cut through verbose explanation.