Fundamental group of the space of maps between surfaces

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Given two closed surfaces M and N, let us consider the function space N^M of all continuous maps from M to N. A question which has been investigated over the past 50 years is the study of the homotopy of the path components of this space, in particular the problem of computing the fundamental group of each path component. This problem has been solved for all closed surfaces except when the target (the surface N) is the projective plane. We will present these results which include contributions of D. Gottlieb, V. L. Hansen, Larmore and E. Thomas. Then we give a solution of the problem when the target surface is the projective space. We divide the discussion into several cases. To illustrate let N_g be the connected sum of g projective plane. If we take the component of the function space $(RP^2)^{N_g}$ which contains the map $f_{2k+1} : N_g \to RP^2$ of absolute degree $2k+1$, then the fundamental group of this component is isomorphic to $Z^{g-1} + Z/2$ for g odd and $Z^{g-1} + Z/4$ for g even. In particular if $g = 1$ and f_{2k+1} is the identity(self-homotopy equivalence of RP^2) then we obtain $Z/2$.