A Good Leaf Order on Simplicial Trees

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Abstract

Using the existence of a good leaf in every simplicial tree, we order the facets of a simplicial tree in order to find combinatorial information about the Betti numbers of its facet ideal. Applications include an Eliahou-Kervaire splitting of the ideal, as well as a refinement of a recursive formula of Hà and Van Tuyl for computing the graded Betti numbers of simplicial trees.

1 Introduction

Given a monomial ideal I in a polynomial ring $R = k[x_1, \dots, x_n]$ over a field k, a **minimal free resolution** of I is an exact sequence of free R-modules

$$0 \to \bigoplus_d R(-d)^{\beta_{p,d}} \to \cdots \to \bigoplus_d R(-d)^{\beta_{0,d}} \to I \to 0$$

of R/I in which $R(-d)$ denotes the graded free module obtained by shifting the degrees of elements in R by d. The numbers $\beta_{i,d}$, which we shall refer to as the i-th $\mathbb{N}$**-graded Betti numbers** of degree d of R/I, are independent of the choice of graded minimal finite free resolution.

Questions about Betti numbers - including when they vanish and when they do not, what bounds they have, how they relate to the base field k and what are the most effective ways to compute them - are of particular interest in combinatorial commutative algebra. Via a method called polarization [Fr], it turns out that it is enough to consider such questions for square-free monomial ideals [GPW]; i.e. a monomial ideal in which the generators are square-free monomials.

To a square-free monomial ideal I one can associate a unique simplicial complex called its facet complex. Conversely, every simplicial complex has a unique monomial ideal assigned to it called its facet ideal [F1]. Simplicial tress [F1] and related structures were developed as a class of simplicial complexes that generalize graph-trees, so that their facet ideals have similar properties to those of edge ideals of graphs discovered in a series of works by Villarreal and his coauthors [V].

This paper offers an order on the monomials generating the facet ideal of a simplicial tree which uses the existence of a "good leaf" in every simplicial tree [HHZ]. This order in itself is

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combinatorially interesting and useful, but it turns out that it also produces a "splitting" [EK] of the facet ideal of a tree which gives bounds on the Betti numbers of the ideal.

Our good leaf order also makes it possible to refine a recursive formula of Hà and Van Tuyl [HV] for computing Betti numbers of facet ideals of simplicial trees, and to apply it to classes of trees with strict good leaf orders. The idea here is that a good leaf order will split an ideal to some extent, and within each one of these split pieces, one can apply Hà and Van Tuyl's formula quite efficiently if the order is strict.

2 Simplicial complexes, trees and forests

Definition 2.1 (simplicial complexes)**.** A **simplicial complex** Δ over a set of vertices $V(\Delta) = \{v_1, \dots, v_n\}$ is a collection of subsets of $V(\Delta)$, with the property that $\{v_i\} \in \Delta$ for all i, and if $F \in \Delta$ then all subsets of F are also in Δ. An element of Δ is called a **face** of Δ, and the **dimension** of a face F of Δ is defined as $|F| - 1$, where $|F|$ is the number of vertices of F. The faces of dimensions 0 and 1 are called **vertices** and **edges**, respectively, and $\dim \emptyset = -1$. The maximal faces of Δ under inclusion are called **facets** of Δ. The dimension of the simplicial complex Δ is the maximal dimension of its facets. A **subcollection** of Δ is a simplicial complex whose facets are also facets of Δ.

A simplicial complex Δ is **connected** if for every pair of facets F, G of Δ, there exists a sequence of facets $F_1, \dots, F_r$ of Δ such that $F_1 = F$, $F_r = G$ and $F_s \cap F_{s+1} \neq \emptyset$ for $1 \leqslant s < r$.

We use the notation $\langle F_1, \dots, F_q \rangle$ to denote the simplicial complex with facets $F_1, \dots, F_q$, and we call it the simplicial complex **generated by** $F_1, \dots, F_q$. By **removing the facet** F_i from Δ we mean the simplicial complex $\Delta \setminus \langle F_i \rangle$ which is generated by $\{F_1, \dots, F_q\} \setminus \{F_i\}$.

Definition 2.2 (Leaf, joint, simplicial trees and forests [F1])**.** A facet F of a simplicial complex Δ is called a **leaf** if either F is the only facet of Δ or for some facet $G \in \Delta \setminus \langle F \rangle$ we have $F \cap H \subseteq G$ for all facets $H \in \Delta \setminus \langle F \rangle$. Such a facet G is called a **joint** of F.

A simplicial complex Δ is a **simplicial forest** if every nonempty subcollection of Δ has a leaf. A connected simplicial forest is called a **simplicial tree**.

It follows easily from the definition that a leaf must always contain at least one **free vertex**, that is a vertex that belongs to no other facet of Δ.

Example 2.3. The facets F_0, F_2 and F_4 are all leaves of the simplicial tree in Figure 1. The first two have F_1 as a joint and F_4 has F_3 as a joint.

Definition 2.4 (Good leaf [Z, CFS])**.** A facet F of a simplicial complex Δ is called a **good leaf** of Δ if F is a leaf of every subcollection of Δ which contains F.

All leaves of the simplicial tree in Figure 1 are good leaves. Figure 2 contains an example of a leaf F in a simplicial tree which is not a good leaf: if we remove the facet G then F is no longer a leaf.

Good leaves were studied in [Z] and then independently in [CFS] (where they were called "reducible leaves"). In both sources the existence of such a leaf in every tree was conjectured but not proved; the proof came later, using incidence matrices.

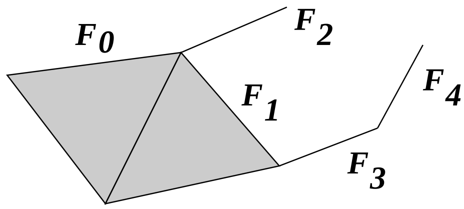

Figure 1: Good leaves

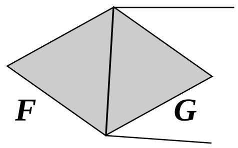

Figure 2: A leaf that is not a good leaf

Theorem 2.5 ([HHZ])**.** *Every simplicial tree contains a good leaf.*

Definition 2.6 (Facet ideal, facet complex [F1])**.** Let Δ be a simplicial complex with vertex set $\{x_1, \ldots, x_n\}$, and let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k with variables corresponding to the vertices of Δ. The **facet ideal** of Δ, denoted by $\mathcal{F}(\Delta)$, is an ideal of R whose generators are monomials, each of which is the products of the variables labeling the vertices of a facet of Δ. Given a square-free monomial ideal I in R, the **facet complex of** I is the simplicial complex whose facets are the set of variables appearing in each monomial generator of I.

Example 2.7. If $I = (xy, yzu, xz)$ is a monomial ideal in $R = k[x, y, z, u]$, its facet complex is the simplicial complex $\Delta = \langle\{x, y\}, \{y, z, u\}, \{x, z\}\rangle$. Similarly I is the facet ideal of Δ.

It is clear from the definition and example that every square-free monomial ideal has a unique facet complex, and every simplicial complex has a unique facet ideal. Because of this one-to-one correspondence we often abuse notation and use facets and monomials interchangeably. For example we say $F \cup G = \mathrm{lcm}(F, G)$ to imply the union of two facets F and G or the least common multiple of two monomials [corresponding to the facets] F and G.

Trees behave well under localization:

Lemma 2.8 (Localization of a tree is a forest [F1])**.** *Let Δ be a simplicial tree with vertices $x_1, \ldots, x_n$, and let I be the facet ideal of Δ in the polynomial ring $R = k[x_1, \ldots, x_n]$ where k is a field. Then for any prime ideal p of R, I_p is the facet ideal of a simplicial forest.*

For a simplicial complex Δ with a facet F, we use the notation $\Delta_{\overline{F}}$ for facet complex of the localization $\mathcal{F}(\Delta)$ at the ideal generated by the the complement of the facet F.

3 Good leaf orders

From its definition it is immediate that a good leaf F_0 of a tree Δ induces an order $F_0, F_1, \dots, F_q$ on the facets of Δ so that

$$F_0 \cap F_1 \supseteq F_0 \cap F_2 \supseteq \cdots \supseteq F_0 \cap F_q.$$

Our goal in this section is to demonstrate that this order can be refined so that Δ is built leaf by leaf starting from the good leaf F_0. In other words, the order can be written so that for $i \leqslant q$, F_i is a leaf of $\Delta_i = \langle F_0, \dots, F_i \rangle$. Such an order on the facets of Δ will be called a **good leaf order** on Δ.

Example 3.1. Let Δ be the simplicial tree in Figure 1. Then F_0 is a good leaf and the labeling of facets $F_0, \dots, F_4$ is a good leaf order on Δ, since $F_0 \cap F_1 \supseteq \cdots \supseteq F_0 \cap F_4$. Note that even though $F_0 \cap F_1 \supseteq F_0 \cap F_2 \supseteq F_0 \cap F_4 \supseteq F_0 \cap F_3$, this latter order $F_0, F_1, F_2, F_4.F_3$ is not a good leaf order since F_3 is not a leaf of Δ.

We show that every simplicial tree (forest) has a good leaf order.

Lemma 3.2. *Suppose $\Delta = \langle F, G, H \rangle$ is a simplicial tree with $F \cap G \not\subseteq H$ and $F \cap H \not\subseteq G$. Then G and H are the leaves of the tree Δ and F is the common joint so that $G \cap H \subseteq F$.*

Proof. If F is a leaf, then either $F \cap G \subseteq H$ or $F \cap H \subseteq G$. Either case is a contradiction, so the two leaves of the tree have to be G and H. If H is a joint of the leaf G then $F \cap G \subseteq H$ which is again a contradiction, so F is the joint of G. Similarly, F is the joint of H, and we have $G \cap H \subseteq F$. $\square$

Proposition 3.3 (First step to build good leaf order)**.** *Let Δ be a simplicial tree with a good leaf F_0 and good leaf order*

$$F_0 \cap F_1 \supseteq F_0 \cap F_2 \supseteq \cdots \supseteq F_0 \cap F_q.$$

Let $1 \leqslant a \leqslant q$ and $0 \leqslant b < a$ and

$$F_0 \cap F_{a-b-1} \supsetneq F_0 \cap F_{a-b} = \cdots = F_0 \cap F_a.$$

Then one of $F_{a-b}, \dots, F_a$ is a leaf of $\langle F_0, \dots, F_a \rangle$.

Proof. Let $\Gamma = \langle F_0, \dots, F_a \rangle$. The subcollection $\Omega = \langle F_0, \dots, F_{a-b-1} \rangle$ of Γ is connected as all facets have nonempty intersection with F_0. If Γ is disconnected, Ω will be contained in one of the connected components of Γ, and there will be another connected component Σ whose facets are from $F_{a-b}, \dots, F_a$. Since Σ is a subcollection of a tree, it must have a leaf, and that leaf will be a leaf of Γ as well. So one of $F_{a-b}, \dots, F_a$ will be a leaf of Γ.

We now assume that Γ is connected and proceed by induction on a to prove our claim. If $a = 1$ then clearly F_1 is a leaf of the tree $\Gamma = \langle F_0, F_1 \rangle$. If $a = 2$ then since $F_2 \cap F_0 \subset F_1$, the facet F_2 must be a leaf with joint F_1.

Now suppose that $a > 2$ and the statement is true up to the $(a-1)$-st step. If $a - b = 1$ then

$$F_0 \cap F_1 = F_0 \cap F_2 = \cdots = F_0 \cap F_a.$$

By [F2] Lemma 4.1 we know that Γ must have two leaves, and so one of the facets $F_1, \dots, F_a$ is a leaf.

We assume that $a - b \geqslant 2$ and neither one of $F_a, \dots, F_{a-b}$ is a leaf of Γ. There are two possible cases.

1. The case $b = 0$. Then $F_0 \cap F_{a-1} \supsetneq F_0 \cap F_a$. If $\Gamma' = \langle F_0, \dots, F_{a-2}, F_a \rangle$ then are two scenarios.

   (a) If Γ' is disconnected, then the facet F_a alone is a connected component of Γ' (since all other facets intersect F_0) and therefore F_a is a leaf of Γ' and $F_a \cap F_i = \emptyset$ for $i = 0, \dots, a-2$. Since Γ is connected, $F_{a-1} \cap F_a \neq \emptyset$, and therefore F_a is a leaf of Γ with joint F_{a-1}.

   (b) If Γ' is connected, we apply the induction hypothesis to the tree Γ' with good leaf F_0. In the ordering of the facets of Γ', F_a can only be at the right end of the sequence (since $F_0 \cap F_{a-2} \supsetneq F_0 \cap F_a$). So F_a is a leaf of Γ' and hence there is a joint $F_j \in \{F_0, \dots, F_{a-2}\}$ such that $F_a \cap F_k \subseteq F_j$ for all $F_k \in \{F_0, \dots, F_{a-2}\}$.
   If F_a is not a leaf of Γ then $F_a \cap F_{a-1} \not\subseteq F_j$. It also follows that $F_a \cap F_j \not\subseteq F_{a-1}$, as otherwise F_{a-1} would be a joint of F_a. Therefore, we can now apply Lemma 3.2 to the tree $\langle F_j, F_{a-1}, F_a \rangle$ to conclude that $F_j \cap F_{a-1} \subseteq F_a$. It follows that

   $$F_0 \cap F_j \cap F_{a-1} \subseteq F_0 \cap F_a \Longrightarrow F_0 \cap F_{a-1} \subseteq F_0 \cap F_a \subsetneq F_{a-1} \cap F_0$$

   which is a contradiction. So F_a has to be a leaf of Γ and we are done.

2. The case $b > 0$. We keep the good leaf F_0 and generate complexes $\Gamma_i = \Gamma \setminus \langle F_i \rangle$ for $i \in \{1, \dots, a\}$. By induction hypothesis each Γ_i has a leaf F_{u_i} where $u_i \in \{a-b, \dots, \hat{i}, \dots, a\}$. Since there are a total of $b+1$ facets that can be leaves of the Γ_i, and there are $a > b+1$ of the complexes Γ_i (recall that we are assuming $a - b \geqslant 2$), we must have $u_i = u_j = u$ for some distinct $i, j \in \{1, \dots, a\}$. Suppose F_{v_i} and F_{v_j} are the joints of F_u in Γ_i and Γ_j, respectively. So we have

   $$\begin{aligned} F_u \cap F_h \subseteq F_{v_i} &\text{ if } h \neq i \\ F_u \cap F_h \subseteq F_{v_j} &\text{ if } h \neq j. \end{aligned} \tag{1}$$

   These two embeddings imply that

   $$\begin{aligned} F_u \cap F_j \subseteq F_{v_i} \cap F_u \subseteq F_{v_j} &\text{ if } v_i \neq j \\ F_u \cap F_i \subseteq F_{v_j} \cap F_u \subseteq F_{v_i} &\text{ if } v_j \neq i. \end{aligned} \tag{2}$$

   Suppose $v_i \neq j$. Then from (1) and (2) we can see that F_u is a leaf of Γ with joint F_{v_j}. Similarly F_u is a leaf of Γ if $v_j \neq i$. So F_u is a leaf of Γ unless $v_i = j$ and $v_j = i$ are the only possible joints for F_u in Γ_i and Γ_j, respectively. In this case (1) turns into

   $$\begin{aligned} F_u \cap F_h \subseteq F_j &\text{ if } h \neq i \\ F_u \cap F_h \subseteq F_i &\text{ if } h \neq j. \end{aligned} \tag{3}$$

Now consider $\Gamma_u = \Delta \setminus \langle F_u \rangle$, which by induction hypothesis must have a leaf F_v with $v \in \{a-b, \ldots, a\} \setminus \{u\}$ and a joint F_t. Since $F_i, F_j \in \Gamma_u$, we must have

$$
\begin{aligned}
F_i \cap F_v \subseteq F_t \text{ if } v \neq i \\
F_j \cap F_v \subseteq F_t \text{ if } v \neq j.
\end{aligned}
\tag{4}
$$

Once again, we consider two cases.

(a) If v can be selected outside $\{i, j\}$, we combine (4) with (3) to get

$$F_u \cap F_v \subseteq F_j \cap F_v \subseteq F_t$$

meaning that F_v is a leaf of Γ.

(b) If v must be in $\{i, j\}$, then the only leaves of Γ_u are F_i and F_j. As $F_0 \in \Gamma_u$ is a good leaf of Δ, one of i and j must be 0, say $j = 0$. But now we have

$$F_u \cap F_j = F_u \cap F_0 \subseteq F_i$$

which together with (3) implies that F_u is a leaf of Γ with joint F_i.

□

Our main theorem is now just a direct consequence of Proposition 3.3, with a bit more added to it.

Theorem 3.4 (Main theorem: good leaf orders)**.** *Let Δ be a simplicial tree with a good leaf F_0. Then there is an order $F_0, F_1, \ldots, F_q$ on the facets of Δ such that*

1. $F_0 \cap F_1 \supseteq F_0 \cap F_2 \supseteq \cdots \supseteq F_0 \cap F_q$*, and*
2. *The facet F_i is a leaf of $\Delta_i = \langle F_0, \ldots, F_i \rangle$ for $0 \leqslant i \leqslant q$.*
3. *The facet F_{i-1} is a either a leaf of Δ_i with the same joint as it has in Δ_{i-1}, or it is the unique joint of F_i in Δ_i, for $1 \leqslant i \leqslant q$.*
4. *$\Delta_i = \langle F_0, \ldots, F_i \rangle$ is connected for $0 \leqslant i \leqslant q$.*

Proof. The good leaf F_0 induces an order on the facets of Δ that satisfies the first property. We need to refine this order to achieve the second property. Let $i \in \{1, \ldots, q\}$. Starting from the beginning, here is how we proceed. For $i \in \{1, \ldots, q\}$ let c_i be the largest nonnegative integer such that $F_i \cap F_0 = F_{i+c_i} \cap F_0$ where $i + c_i \leqslant q$.

Set $i = 1$.

Step 1 If $c_i = 0$ then set $i := i + 1$ and go back to Step 1.

Step 2 If $c_i > 0$ then we reorder $F_i, \dots, F_{i+c_i}$ as follows. By Proposition 3.3 there is a leaf $F_{\ell_{c_i}} \in \{F_i, \dots, F_{i+c_i}\}$ of $\Gamma = \langle F_0, \dots, F_{i+c_i} \rangle$. Applying the same proposition again there is a leaf $F_{\ell_{c_i - 1}} \in \{F_i, \dots, F_{i+c_i}\} \setminus \{F_{\ell_{c_i}}\}$ of $\Gamma \setminus \langle F_{\ell_{c_i}} \rangle$. We continue this way $c_i + 1$ times and in the end we have a sequence

$$F_{\ell_0}, F_{\ell_1}, \dots, F_{\ell_{c_i}}$$

which is a reordering of the facets $F_i, \dots, F_{i+c_i}$ that satisfies both properties (1) and (2) in the statement of the theorem. We relabel $F_i, \dots, F_{i+c_i}$ with this new order and set $i := i + c_i + 1$.

Step 3 If $i > q$ we stop and otherwise we go back to Step 1.

At the end of this algorithm, the facets of Δ have the desired order.

To prove the third part of the theorem, note that as F_{i-1} is a leaf in Δ_{i-1}, it has a set of free vertices in Δ_{i-1} which we call A. There are two scenarios.

- If $F_i \cap A \neq \emptyset$, then F_{i-1} has to be the unique joint of F_i in Δ_i, as no other facet of Δ_i would contain any element of A.
- If $F_i \cap A = \emptyset$, then $F_i \cap F_{i-1} \subseteq \Delta_{i-2} \cap F_{i-1} \subseteq F_\alpha$, where F_α is the joint of F_{i-1} in Δ_{i-1}. Therefore, F_{i-1} is a leaf of Δ_i.

Finally to see that Δ_i is connected for every i, we consider two situations.

1. $F_i \cap F_0 \neq \emptyset$. In this case Δ_i is connected as all facets of Δ_i intersect F_0.
2. $F_i \cap F_0 = \emptyset$. If $i = q$ then $\Delta_i = \Delta$ which is connected. Now we assume that i is the smallest index with $F_i \cap F_0 = \emptyset$, and $c_i > 0$, and we consider how $\Delta_i, \dots, \Delta_q = \Delta$ are built in Step 2. We start from Δ, and pick a leaf for Δ from $F_i, \dots, F_q$. We call this facet F_q and we know already that $\Delta_q = \Delta$ must be connected. To pick Δ_{q-1} we remove the leaf F_q from Δ, and so Δ_{q-1} has to be connected. To build Δ_{q-2} we again remove a leaf from Δ_{q-1}, which forces Δ_{q-2} to be connected, and so on until we reach Δ_i, which by the same reasoning has to be connected.

□

4 The effect of good leaf orders on resolutions

Recall that for a monomial ideal I, the notation $\mathcal{G}(I)$ denotes the unique minimal monomial generating set for I.

Definition 4.1 (Splitting [EK]). A monomial ideal I is called **splittable** if one can write $I = J + K$ for two nonzero monomial ideals J and K, such that

1. $\mathcal{G}(I)$ is the disjoint union of $\mathcal{G}(J)$ and $\mathcal{G}(K)$;

2. There is a *splitting function* $\mathcal{G}(J \cap K) \to \mathcal{G}(J) \times \mathcal{G}(K)$ taking each $w \in \mathcal{G}(J \cap K)$ to $(\phi(w), \psi(w))$ satisfying

   (a) For each $w \in \mathcal{G}(J \cap K)$, $w = \mathrm{lcm}(\phi(w), \psi(w))$

   (b) For each $S \subseteq \mathcal{G}(J \cap K)$, $\mathrm{lcm}(\phi(S))$ and $\mathrm{lcm}(\psi(S))$ strictly divide $\mathrm{lcm}(S)$.

If a monomial ideal is splittable, then its Betti numbers can be broken down into those of sub-ideals.

Theorem 4.2 ([EK, Fa])**.** *If I is a monomial ideal with a splitting $I = J + K$, then for all $i, j \geqslant 0$*

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K).$$

Our next observation is that a goof leaf order on a simplicial tree provides a basic splitting of its facet ideal.

Theorem 4.3 (Splitting using a good leaf order)**.** *If I is the facet ideal of a simplicial tree Δ with a good leaf F_0 and good leaf order*

$$F_0 \cap F_1 \supseteq F_0 \cap F_2 \supseteq \ldots \supseteq F_0 \cap F_t \supsetneq F_0 \cap F_{t+1} = \ldots = F_0 \cap F_q = \emptyset$$

and $J = (F_0, \ldots, F_t)$ and $K = (F_{t+1}, \ldots, F_q)$, then $I = J + K$ is a splitting of I.

Proof. It is clear that $I = J+K$. We number the vertices of $F_0, \ldots, F_t$ in some order as $x_1, \ldots, x_m$. We will build ϕ and ψ as in Definition 4.1. Suppose $L \in \mathcal{G}(J \cap K)$. Then there are facets F_i and F_j such that $i \leqslant t < j$ such that $L = \mathrm{lcm}(F_i, F_j)$. Of all choices of such F_i we pick one minimal with respect to lex order and call it G_L, and there is only one choice for F_j (since each F_j adds one or more new vertices to the sequence $F_0, \ldots, F_{j-1}$); call this facet H_L. So we have $L = \mathrm{lcm}(G_L, H_L)$. Let $\phi(L) = G_L$ and $\psi(L) = H_L$ so that we have a map

$$\mathcal{G}(J \cap K) \to \mathcal{G}(J) \times \mathcal{G}(K) L \to (\phi(L), \psi(L)) = (G_L, H_L).$$

We only need to show that Condition (b) in Definition 4.1 holds. Suppose $S = \{L_1, \ldots, L_r\} \subseteq \mathcal{G}(J \cap K)$. Suppose, as before, for each i we can write $L_i = \mathrm{lcm}(G_{L_i}, H_{L_i}) = G_{L_i} \cup H_{L_i}$ where $G_{L_i} \in \mathcal{G}(K)$ and $H_{L_i} \in \mathcal{G}(K)$. We need to show

1. $G_{L_1} \cup \cdots \cup G_{L_r} \subsetneq L_1 \cup \cdots \cup L_r$.

   This is clear since each of the L_i contains vertices that are in $\mathcal{G}(K)$ but not in $\mathcal{G}(J)$.

2. $H_{L_1} \cup \cdots \cup H_{L_r} \subsetneq L_1 \cup \cdots \cup L_r$

   Each of the L_i has a nonempty intersection with F_0, but $H_{L_i} \cap F_0 = \emptyset$, which makes the inclusion above strict.

So we have shown that we have a splitting which completes the proof. □

As a result, we can use good leaf orders to bound invariants related to resolutions of trees. The following statement is a direct application of theorems 4.2 and 4.3.

Corollary 4.4 (Bounds on Betti numbers of trees)**.** *Suppose Δ is a simplicial tree that can be partitioned into subcollections $\Delta_0, \ldots, \Delta_s$, each of which is a tree, and such that for each $i = 0, \ldots, s$, setting $a_0 = 0$ we have:*

1. $\Delta_i = \langle F_{a_i}, F_{a_i+1}, \ldots, F_{a_{i+1}-1} \rangle$ *with good leaf* F_{a_i}.
2. $F_{a_i+1} \cap F_{a_i} \supseteq \ldots \supseteq F_{a_{i+1}-1} \cap F_{a_i} \neq \emptyset$ *is a good leaf order on* Δ_i*;*
3. $F_{a_i} \cap F_j = \emptyset$ *for* $j \geqslant a_{i+1}$.

Then

$$\beta_{i,j}(\Delta) \geqslant \beta_{i,j}(\Delta_0) + \cdots + \beta_{i,j}(\Delta_s).$$

In particular

$$\textit{projdim}(\Delta) \geqslant \max\{\textit{projdim}(\Delta_0), \ldots, \textit{projdim}(\Delta_s)\}$$

and

$$\textit{reg}(\Delta) \geqslant \max\{\textit{reg}(\Delta_0), \ldots, \textit{reg}(\Delta_s)\}.$$

4.1 Recursive calculations of Betti numbers

In [HV] Hà and Van Tuyl used Eliahou-Kervaire splittings to reduce the computation of the Betti numbers of a given simplicial forest to that of smaller ones. Our goal here is to show that their formula can be refined certain cases and be used to compute the Betti numbers of a given simplicial tree in terms of intersections of the faces. The method used is essentially a repeated application of a splitting formula due to Hà and Van Tuyl [HV] to a good leaf order on a given tree, along with an argument that, at every stage, we know what the next splitting to consider should be.

Definition 4.5 ([HV] Definition 5.1)**.** Let F be a facet of a simplicial complex Δ. The **connected component of** F **in** Δ, denoted $\text{conn}_\Delta(F)$, is defined to be the connected component of Δ containing F. If $\text{conn}_\Delta(F) = \langle G_1, \ldots, G_p \rangle$, then we define the **reduced connected component of** F **in** Δ, denoted by $\overline{\text{conn}}_\Delta(F)$, to be the simplicial complex whose facets are a subset of $\{G_1 \setminus F, \ldots, G_p \setminus F\}$, chosen so that if there exist G_i and G_j such that $\emptyset \neq G_i \setminus F \subseteq G_j \setminus F$, then we shall disregard the bigger facet $G_j \setminus F$ in $\overline{\text{conn}}_\Delta(F)$.

Note that in the Definition 4.5, $\overline{\text{conn}}_\Delta(F)$ is the localization of $\text{conn}_\Delta(F)$ at the ideal generated by the complement of the facet F. Therefore if Δ is a tree then $\overline{\text{conn}}_\Delta(F)$ is always a forest ([F1]). Hà and Van Tuyl ([HV] Lemma 5.7) prove this directly in their paper.

A facet F of Δ is called a **splitting facet of** Δ if $\mathcal{F}(\Delta) = (F) + \mathcal{F}(\Delta \setminus \langle F \rangle)$ is a splitting of $\mathcal{F}(\Delta)$ (here we are thinking of F as a monomial).

Theorem 4.6 ([HV] Theorem 5.5)**.** *If F is a splitting facet of a simplicial complex Δ, then for all $i \geqslant 1$ and $j \geqslant 0$ we have*

$$\beta_{i,j}(\mathcal{F}(\Delta)) = \beta_{i,j}(\mathcal{F}(\Delta \setminus \langle F \rangle)) + \sum_{l_1=0}^{i} \sum_{l_2=0}^{j-|F|} \beta_{l_1-1,l_2}(\mathcal{F}(\overline{\textit{conn}}_\Delta(F))\beta_{i-l_1-1,j-|F|-l_2}(\mathcal{F}(\Delta \setminus \textit{conn}_\Delta(F)). \tag{5}$$

So now the question is what is a good choice for a splitting facet. In their paper ([HV] Theorem 5.6) Hà and Van Tuyl show that a leaf of a simplicial complex is a splitting facet. Their proof (which we repeat below for the sake of having a written proof for our claim) in fact only requires the facet to have a free vertex.

Proposition 4.7. *Let Δ be a simplicial complex. If F is a facet of Δ with a free vertex, then F is a splitting facet of Δ.*

Proof. We need to show that $\mathcal{F}(\Delta) = (F) + \mathcal{F}(\Delta \setminus \langle F \rangle)$ is a splitting. Let $J = (F)$ and $K = \mathcal{F}(\Delta \setminus \langle F \rangle)$. Without loss of generality, we may assume that $F = x_1 \dots x_l$. We shall construct a splitting function $s : \mathcal{G}(J \cap K) \to \mathcal{G}(J) \times \mathcal{G}(K)$ for $\mathcal{F}(\Delta)$. Suppose $L \in \mathcal{G}(J \cap K)$. Let $\mathcal{M}_L = \{G \in \mathcal{G}(K) \mid \mathrm{lcm}(F, G) = L\}$. For each $G \in \mathcal{M}_L$, we order the elements of $G \cap F$ by the increasing order of their indices and view $G \cap F$ as an ordered word of the alphabet $\{x_1, \dots, x_l\}$. Let $G_L \in \mathcal{M}_L$ be such that $G_L \cap F$ is minimal with respect to the lexicographic word order. Clearly, G_L is uniquely determined by L. Our splitting function s is defined as follows. For each $L \in \mathcal{G}(J \cap K)$,

$$s(L) = (\phi(L), \psi(L)) = (F, G_L).$$

We need to verify that s satisfies conditions (a) and (b) of Definition 4.1. Indeed, condition (a) follows obviously from the definition of the function s. Suppose $S \subseteq \mathcal{G}(J \cap K)$. The facet F has a free vertex u, and therefore u does not divide $\mathrm{lcm}(\psi(S))$. Yet, since u is in F, u divides $\mathrm{lcm}(S)$. Thus, $\mathrm{lcm}(\psi(S))$ strictly divides $\mathrm{lcm}(S)$. On the other hand, it is also clear that for any $G \in \mathcal{G}(K)$, F strictly divides $\mathrm{lcm}(F, G)$, so $\mathrm{lcm}(\phi(S)) = F$ strictly divides $\mathrm{lcm}(S)$. The statement is proved. ☐

We use the convention that for any ideal I

$$\beta_{-1,j}(I) = \begin{cases} 1 & j = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Suppose we have a simplicial tree Δ with good leaf order described as in Theorem 3.4. We apply (5) to $\Delta = \langle F_0, \dots, F_q \rangle$ peeling off leaves in the following order: $F_q, F_{q-1}, \dots, F_0$.

Suppose we are in step u, peeling off the leaf F_u from the tree $\Delta_u = \langle F_0, \dots, F_u \rangle$. Then $\mathrm{conn}_{\Delta_u}(F_u) = \Delta_u$ and so $\mathcal{F}(\Delta_u \setminus \mathrm{conn}_{\Delta_u}(F_u)) = 0$ and therefore

$$\beta_{a,b}(\mathcal{F}(\Delta_u \setminus \mathrm{conn}_{\Delta_u}(F_u)) = \begin{cases} 1 & a = -1,\ b = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Applying this to (5), we solve $i - l_1 - 1 = -1$ and $j - |F_u| - l_2 = 0$ to find $l_1 = i$ and $l_2 = j - |F_u|$.

Moreover, we have $\overline{\mathrm{conn}}_{\Delta_u}(F_u) = (\Delta_{u-1})_{\overline{F_u}}$, that is Δ_{u-1} localized at the ideal generated by the complement of the facet F_u using notation as in Lemma 2.8. So (5) turns into

$$\begin{aligned}\beta_{i,j}(\mathcal{F}(\Delta)) &= \beta_{i,j}(\mathcal{F}(\Delta_{q-1})) + \beta_{i-1,j-|F_q|}(\mathcal{F}((\Delta_{q-1})_{\overline{F_q}}))\\ &= \beta_{i,j}(\mathcal{F}(\Delta_{q-2})) + \beta_{i-1,j-|F_{q-1}|}(\mathcal{F}((\Delta_{q-2})_{\overline{F_{q-1}}})) + \beta_{i-1,j-|F_q|}(\mathcal{F}((\Delta_{q-1})_{\overline{F_q}}))\\ &\vdots\\ &= \beta_{i,j}(\mathcal{F}(\langle F_0\rangle)) + \sum_{u=1}^{q}\beta_{i-1,j-|F_u|}(\mathcal{F}((\Delta_{u-1})_{\overline{F_u}}))\end{aligned}$$

Note that we did not use the fact that F_0 us a good leaf here, just that each F_u is a leaf if Δ_u. We have therefore justified the following statement.

Proposition 4.8. *Let Δ be a simplicial tree with a good leaf order $F_0, F_1, \ldots, F_q$ such that each F_u is a leaf of $\Delta_u = \langle F_0, \ldots, F_u\rangle$ for $u \leqslant q$. Then for all $i \geqslant 1$ and $j \geqslant 0$*

$$\beta_{i,j}(\mathcal{F}(\Delta)) = \beta_{i,j}(\mathcal{F}(\langle F_0\rangle)) + \sum_{u=1}^{q}\beta_{i-1,j-|F_u|}(\mathcal{F}((\Delta_{u-1})_{\overline{F_u}})). \tag{7}$$

By introducing and appropriate "δ" function we can say

$$\beta_{i,j}(\mathcal{F}(\langle F_0\rangle)) = \delta_{(i,j),(0,|F_0|)} = \begin{cases} 1 & i=0,\ j=|F_0| \\ 0 & \text{otherwise}. \end{cases} \tag{8}$$

So now we focus on the structure of $(\Delta_{u-1})_{\overline{F_u}}$. The main point that we would like to make is that $(\Delta_{u-1})_{\overline{F_u}}$ behaves well, in other words, it satisfies the same kind of inclusion sequence enforced in Theorem 3.4, and the same "leaf-peeling" property. Note that though F_0 need not even survive the localization, its role is that of a virtual glue that forces facets to always stick together and have an appropriate order.

Proposition 4.9. *Let Δ be a simplicial tree with a good leaf F_0 and good leaf order*

$$F_0 \cap F_1 \supsetneq F_0 \cap F_2 \supsetneq \cdots \supsetneq F_0 \cap F_q.$$

Suppose $u \in \{1, \ldots, q\}$ and $\Delta_u = \langle F_1, \ldots, F_u\rangle$, and suppose $(\Delta_{u-1})_{\overline{F_u}}$ has facets $F_{a_1} \setminus F_u, \ldots,$ $F_{a_s} \setminus F_u$ with $0 \leqslant a_1 < \ldots < a_s \leqslant u-1$. Then

1. *$a_s = u-1$,*
2. *$(F_0 \cap F_{a_1}) \setminus F_u \supsetneq \ldots \supsetneq (F_0 \cap F_{a_s}) \setminus F_u$,*
3. *$(\Delta_{u-1})_{\overline{F_u}}$ is a simplicial tree,*
4. *If F_v is a joint of F_u in Δ_u then $F_v \setminus F_u \in (\Delta_{u-1})_{\overline{F_u}}$,*
5. *$F_{u-1} \setminus F_u$ has a free vertex in $(\Delta_{u-1})_{\overline{F_u}}$.*

Proof. To prove 1, suppose there is an $i < u-1$ such that $(F_i \setminus F_u) \subset (F_{u-1} \setminus F_u)$. By assumption there exists $y \in (F_0 \cap F_i) \setminus (F_0 \cap F_{u-1})$. As $(F_0 \cap F_{u-1}) \supset (F_0 \cap F_u)$, it follows that $y \in (F_i \setminus F_u)$ and $y \notin (F_{u-1} \setminus F_u)$, which contradicts the inclusion $(F_i \setminus F_u) \subset (F_{u-1} \setminus F_q)$.

The strict inclusions in 2 follow from the same observation, that for every i there is always an element in $F_0 \cap F_{a_i}$ which is not in $F_{a_{i+1}}$ or F_u.

Since $(\Delta_{u-1})_{\overline{F_u}}$ is a localization of the tree Δ_{u-1}, it is clear that it is a forest, and by 2, since $(F_0 \cap F_{a_s}) \setminus F_u \neq \emptyset$, it must be connected and therefore a simplicial tree. This settles 3.

For 4, suppose for some $j < u$ we have $F_j \setminus F_u \subseteq F_v \setminus F_u$. Then we will have

$$F_j = (F_j \cap F_u) \cup (F_j \setminus F_u) \subseteq F_v$$

which implies that $F_j = F_v$.

Finally to prove 5 we use induction on u. If $u = 1$ or 2, then $(\Delta_{u-1})_{\overline{F_u}}$ will have one or two facets, and in each case $F_{u-1} \setminus F_u$ clearly must have a free vertex. If $u = 3$ then F_2 is a leaf of Δ_2 with a joint F_i for some $i < 2$. If $F_i \setminus F_3 \in (\Delta_2)_{\overline{F_3}}$, then it acts as a joint of $F_2 \setminus F_3$ so $F_2 \setminus F_3$ is a leaf and must therefore have a free vertex. If $F_i \setminus F_3 \notin (\Delta_2)_{\overline{F_3}}$, then $(\Delta_2)_{\overline{F_3}}$ has at most two facets including $F_2 \setminus F_3$, each of which must have a free vertex. This settles the base cases for induction.

Now suppose $u \geqslant 4$ and $F_{u-1} \setminus F_u$ has no free vertex in $(\Delta_{u-1})_{\overline{F_u}}$.

By induction hypothesis, if we consider $\Gamma = \Delta \setminus \langle F_{u-2} \rangle$, then $F_{u-1} \setminus F_u$ will have a free vertex x in $(\Gamma_{u-1})_{\overline{F_u}}$. If x is not a free vertex in $(\Delta_{u-1})_{\overline{F_u}}$, then for some $j < u-1$ we have $x \in F_j \setminus F_u \in (\Delta_{u-1})_{\overline{F_u}}$ and $F_j \setminus F_u \notin (\Gamma_{u-1})_{\overline{F_u}}$. The only possible such index j is $j = u-2$. In other words, $x \in F_{u-1} \cap F_{u-2}$ and $x \notin F_i$ for any other $i \leqslant u$.

Similarly, if we remove F_{u-3} from Δ we will find a vertex $y \in F_{u-1} \cap F_{u-3}$ and $y \notin F_i$ for any other $i \leqslant u$.

By Lemma 3.2, we must then have $F_{u-3} \cap F_{u-2} \subseteq F_{u-1}$. Intersecting both sides with F_0 we obtain

$$F_{u-1} \cap F_0 \subseteq F_{u-2} \cap F_0 = F_{u-3} \cap F_{u-2} \cap F_0 \subseteq F_{u-1} \cap F_0$$

which means that $F_{u-1} \cap F_0 = F_{u-2} \cap F_0$; a contradiction. □

Proposition 4.9 now allows us to continue solving (7) by applying Theorem 4.6 once again, since we have a splitting facet for each $(\Delta_{u-1})_{\overline{F_u}}$. Consider the tree Δ as described above with the good leaf order described in Theorem 3.4 and for some $u \in \{1, \ldots, q\}$, let $(\Delta_{u-1})_{\overline{F_u}} = \langle F_{a_1} \setminus F_u, \ldots, F_{a_s} \setminus F_u \rangle$ where $0 \leqslant a_1 < \ldots < a_s < u$. By Proposition 4.9 $(\Delta_{u-1})_{\overline{F_u}}$ is a simplicial tree with an order of the facets induced by the good leaf order of Δ, and with splitting facet $F_{a_s} \setminus F_u$.

We continue in the same spirit. Let $u_1 = u, u_2 = a_s$ and

$$\mathcal{C}_{u_1,u_2} = ((\Delta_{u_1-1})_{F_{u_1}})_{F_{u_2} \setminus F_{u_1}} = \langle F_{d_1} \setminus (F_{u_1} \cup F_{u_2}), \ldots, F_{d_w} \setminus (F_{u_1} \cup F_{u_2}) \rangle$$

where $0 \leqslant d_1 < \ldots < d_w < u_2 < u_1$.

Similarly, we can build $\mathcal{C}_{u_1,\ldots,u_m}$ which is the localization of

$$C_{u_1,\ldots,u_{m-1}} = \langle F_{c_1} \setminus (F_{u_1} \cup \ldots \cup F_{u_{m-1}}), \ldots, F_{c_r} \setminus (F_{u_1} \cup \ldots \cup F_{u_{m-1}}) \rangle \qquad (9)$$

at the ideal generated by the complement of the facet $F_{u_m} \setminus (F_{u_1} \cup \ldots \cup F_{u_{m-1}})$ where $u_m = c_r$. So we have

$$\mathcal{C}_{u_1,\ldots,u_m} = \langle F_{b_1} \setminus (F_{u_1} \cup \ldots \cup F_{u_m}), \ldots, F_{b_t} \setminus (F_{u_1} \cup \ldots \cup F_{u_m}) \rangle \tag{10}$$

where $b_1, \ldots, b_t \in \{c_1, \ldots, c_{r-1}\}$, and

$$0 \leqslant b_1 < b_2 < \ldots < b_t < c_r = u_m < u_{m-1} < \ldots < u_1.$$

Proposition 4.10. *Let Δ be a simplicial tree with a good leaf F_0 and good leaf order*

$$F_0 \cap F_1 \supsetneq F_0 \cap F_2 \supsetneq \cdots \supsetneq F_0 \cap F_q.$$

With notation as in (9) and (10) above, we have

1. $b_t = c_{r-1}$,
2. $(F_0 \cap F_{b_1}) \setminus (F_{u_1} \cup \ldots \cup F_{u_m}) \supsetneq \ldots \supsetneq (F_0 \cap F_{b_t}) \setminus (F_{u_1} \cup \ldots \cup F_{u_m})$,
3. $\mathcal{C}_{u_1,\ldots,u_m}$ *is a simplicial tree,*
4. $F_{b_t} \setminus (F_{u_1} \cup \ldots \cup F_{u_m})$ *has a free vertex in* $\mathcal{C}_{u_1,\ldots,u_m}$ *and is therefore a splitting facet of* $\mathcal{C}_{u_1,\ldots,u_m}$.

Proof. Let $A = F_{u_1} \cup \ldots \cup F_{u_m}$. To show 1, suppose there is an $i < r-1$ such that $F_{c_i} \setminus A \subset F_{c_{r-1}} \setminus A$. By the strict inclusions assumed there exists $y \in (F_0 \cap F_{c_i}) \setminus (F_0 \cap F_{c_{r-1}})$. As

$$F_0 \cap F_{c_{r-1}} \supsetneq F_0 \cap F_{u_m} \supsetneq \ldots \supsetneq F_0 \cap F_{u_1},$$

it follows that $y \in F_{c_i} \setminus A$ and $y \notin F_{c_{r-1}} \setminus A$, which is a contradiction.

For 2 it is easy to see that

$$(F_0 \cap F_{b_1}) \setminus A \supseteq \ldots \supseteq (F_0 \cap F_{b_t}) \setminus A.$$

To show that these inclusions are strict pick $1 \leqslant i < j < t$, we know that

$$F_0 \cap F_{b_i} \supsetneq F_0 \cap F_{b_j} \supsetneq F_0 \cap F_{u_m} \supsetneq F_0 \cap F_{u_{m-1}} \supsetneq \ldots \supsetneq F_0 \cap F_{u_1},$$

and therefore there exists $y \in (F_{b_i} \cap F_0) \setminus (F_{b_j} \cup F_{u_1} \cup \ldots \cup F_{u_m})$, which means that $y \in (F_0 \cap F_{b_i}) \setminus (F_{u_1} \cup \ldots \cup F_{u_m})$ and $y \notin (F_0 \cap F_{b_j}) \setminus (F_{u_1} \cup \ldots \cup F_{u_m})$, proving 2.

Suppose $\Omega = \langle F_{\omega_0}, F_{\omega_1}, \ldots, F_{\omega_p} \rangle$ is the subcollection of Δ consisting of those facets that are not contained in A with

$$0 = \omega_0 < \omega_1 < \ldots < \omega_p.$$

Because of the strict good leaf order Ω is a connected forest and hence a tree.

We claim that $C_{u_1,\ldots,u_m}$ is the localization of the tree Ω at the ideal generated by $\overline{A}$. This follows from two observations. One is that if at the ith step when building $C_{u_1,\ldots,u_m}$ there are facets $F_\alpha, F_\beta \in \Delta$ not containing $F_{u_1} \cup \ldots \cup F_{u_i}$, then F_α, F_β do not contain A and therefore are also facets of Ω. Moreover if $F_\alpha \setminus (F_{u_1} \cup \ldots \cup F_{u_i}) \subseteq F_\beta \setminus (F_{u_1} \cup \ldots \cup F_{u_i})$, then $F_\alpha \setminus A \subseteq F_\beta \setminus A$ and therefore we can conclude that $C_{u_1,\ldots,u_m}$ is a localization Ω and $\{b_1 \ldots b_t\} \subseteq \{\omega_0, \ldots, \omega_p\}$.

So $C_{u_1,\ldots,u_m}$ must be a forest, and since it is connected by 2, it must be a simplicial tree. This settles 3.

By the discussion above we can assume $\omega_p = b_t$ and we will still have $C_{u_1,\ldots,u_m}$ is a localization of Ω. Also note that $F_0 = F_{\omega_0}$ is a good leaf of Ω with a strict good leaf order induced by that on Δ.

To prove 4 we use induction on p. If $p = 1$ or 2 then $C_{u_1,\ldots,u_m}$ will have one or two facets, and in each case $F_{b_t} \setminus A$ clearly must have a free vertex. If $p = 3$ then F_{ω_2} is a leaf of $\Omega_{\omega_2} = \langle F_{\omega_0}, F_{\omega_1}, F_{\omega_2} \rangle$ with a joint F_{w_i} for some $i < 2$. If $F_{\omega_i} \setminus A \in C_{u_1,\ldots,u_m}$, then it acts as a joint of $F_{\omega_2} \setminus A$ so $F_{\omega_2} \setminus A$ is a leaf and must therefore have a free vertex. If $F_{\omega_i} \setminus A \notin C_{u_1,\ldots,u_m}$, then $C_{u_1,\ldots,u_m}$ has at most two facets including $F_{\omega_2} \setminus A$ each of which must have a free vertex. This settles the base cases for induction.

Now suppose $p \geqslant 4$ and $F_{b_t} \setminus A$ has no free vertex in $C_{u_1,\ldots,u_m}$.

By the induction hypothesis, if we consider $\Gamma = \Omega \setminus \langle F_{\omega_{p-1}} \rangle$ then F_{ω_p} will have a free vertex x in $\Gamma_{\overline{A}}$. If x is not a free vertex in $\Gamma_{\overline{A}}$ then $x \in F_{\omega_{p-1}} \setminus A \in \Gamma_{\overline{A}}$. In other words, $x \in F_{\omega_p} \cap F_{\omega_{p-1}}$ and $x \notin F_{\omega_i}$ for any other $i \leqslant p$.

Similarly, if we remove $F_{\omega_{p-2}}$ from Ω we will find a vertex $y \in F_{\omega_p} \cap F_{\omega_{p-2}}$ and $y \notin F_i$ for any other $i \leqslant p$.

By Lemma 3.2, we must then have $F_{\omega_{p-2}} \cap F_{\omega_{p-1}} \subseteq F_{\omega_p}$. Intersecting both sides with F_0 we obtain

$$F_{\omega_p} \cap F_0 \subseteq F_{\omega_{p-1}} \cap F_0 = F_{\omega_{p-2}} \cap F_{\omega_{p-1}} \cap F_0 \subseteq F_{\omega_p} \cap F_0$$

which means that $F_{\omega_p} \cap F_0 = F_{\omega_{p-1}} \cap F_0$; a contradiction. This proves 4 and we are done. $\square$

Proposition 4.10 replaces Proposition 4.9 as a more general version. Back to (7), we start computing Betti numbers of $\mathcal{F}(\Delta)$ for a given tree Δ with good leaf F_0 and strict good leaf order

$$F_0 \cap F_1 \supsetneq F_0 \cap F_2 \supsetneq \cdots \supsetneq F_0 \cap F_q.$$

The formula

$$\beta_{i,j}(\mathcal{F}(\Delta)) = \beta_{i,j}(\mathcal{F}(\langle F_0 \rangle)) + \sum_{u=1}^{q} \beta_{i-1,j-|F_u|}(\mathcal{F}((\Delta_{u-1})_{\overline{F_u}}))$$

becomes recursive, since in each step after localization we again have a simplicial tree with a strict induced order on the facets where the last facet remaining is a splitting facet.

To close, we apply the formula to examine some low Betti numbers.

Let $i = 0$. By (7) and (6) we have

$$\beta_{0,j}(\mathcal{F}(\Delta)) = \sum_{u=0}^{q} \delta_{j,|F_u|}.$$

Let $i \geqslant 1$. Because of (7) and (8) we can write

$$\beta_{i,j}(\mathcal{F}(\Delta)) = \sum_{u=1}^{q} \beta_{i-1,j-|F_u|}(\mathcal{F}((\Delta_{u-1})_{\overline{F_u}})) \tag{11}$$

From Proposition 4.10 and (11) we can see that we need the generators of each Δ_u in order to produce a formula for the first graded Betti numbers. To this end, we start from $\Delta_u = \langle F_0, \dots, F_u \rangle$ so that

$$\begin{aligned}
(\Delta_{u-1})_{\overline{F_u}} &= \langle F_i \setminus F_u \mid 0 \leqslant i < u \text{ and } (F_j \setminus F_u) \not\subseteq (F_i \setminus F_u) \text{ for } j \neq i \rangle \\
&= \langle F_i \setminus F_u \mid 0 \leqslant i < u \text{ and } \frac{\mathrm{lcm}(F_j, F_u)}{F_u} \nmid \frac{\mathrm{lcm}(F_i, F_u)}{F_u} \text{ for } j \neq i \rangle \\
&= \langle F_i \setminus F_u \mid 0 \leqslant i < u \text{ and } \mathrm{lcm}(F_j, F_u) \nmid \mathrm{lcm}(F_i, F_u) \text{ for } j \neq i \rangle
\end{aligned}$$

So we can make our "delta-function" to have the lcm condition built into it. We define

$$\delta_{a,(b,c)} = \begin{cases} 1 & a = |F_b|,\ \mathrm{lcm}(F_d, F_c) \nmid \mathrm{lcm}(F_b, F_c) \text{ for } 0 \leqslant d < c \\ 0 & \text{otherwise} \end{cases}$$

So (11) becomes

$$\begin{aligned}
\beta_{1,j}(\mathcal{F}(\Delta)) &= \sum_{u=1}^{q} \beta_{0,j-|F_u|}(\mathcal{F}((\Delta_{u-1})_{\overline{F_u}})) \\
&= \sum_{u=1}^{q} \sum_{F \text{ facet of } (\Delta_{u-1})_{\overline{F_u}}} \delta_{j-|F_u|,|F|} \\
&= \sum_{u=1}^{q} \sum_{v=0}^{u-1} \delta_{j-|F_u|,(v,u)}
\end{aligned}$$

By building appropriate delta functions, one can continue in this manner to build further Betti numbers based on the lcms of the facets.

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