Toric Ideals of Weighted Oriented Graphs

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January 20, 2023

Toric Ideals

Throughout this talk, k will be a field.

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Definition

A *toric ideal* is the kernel of a monomial map between polynomial rings:

$$\varphi : k[x_1, \ldots, x_n] \to k[y_1, \ldots, y_s]$$

$$\varphi(x_i) = m_i,$$

where each m_i is a monomial.

Toric Ideals - Example

$$\varphi : k[x, y, z, w] \to k[s, t]$$

$$\varphi(x) = t, \;\; \varphi(y) = st, \;\; \varphi(z) = s^2 t, \;\; \varphi(w) = s^3 t$$

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$$\ker(\varphi) = (xw - yz, y^2 - xz, z^2 - yw)$$

Toric Ideals - Basic Facts

- Toric ideals are generated by binomials.
- Multigraded using the domain polynomial ring
- Toric varieties are defined by toric ideals, but generally one requires that a toric variety be normal which is more restrictive.
- Toric ideals are used in integer programming problems, algebraic statistics, algebraic models of chemical networks, etc.

Toric Ideals of Graphs

Given a (finite, simple) graph $G = (V, E)$ on vertex set $V = \{x_1, \ldots, x_n\}$ and edge set $E = \{e_1, \ldots, e_t\}$ we define a map

$$\begin{aligned} \varphi : k[e_1, \ldots, e_t] &\to k[x_1, \ldots, x_n] \\ e_i &\mapsto x_{i_1} x_{i_2} \end{aligned}$$

where $e_i = \{x_{i_1}, x_{i_2}\}$.

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where $e_i = \{x_{i_1}, x_{i_2}\}$.

The kernel of φ is the *toric ideal of the graph* G which we denote I_G.

Toric Ideals of Graphs

In a graph $G = (V, E)$, a even closed walk is a sequence of vertices $v_0, v_1, \ldots, v_{2p}$ where

- $\{v_{i-1}, v_i\}$ is an edge in G for all $1 \leq i \leq 2p$
- $v_0 = v_{2p}$.

Given an even closed walk $W = (v_0, v_1, \ldots, v_{2p})$ with $\{v_{i-1}, v_i\} = e_i$ we define a binomial

$$f_W = \prod_{i=1}^{p} e_{2i} - \prod_{i=1}^{p} e_{2i-1}.$$

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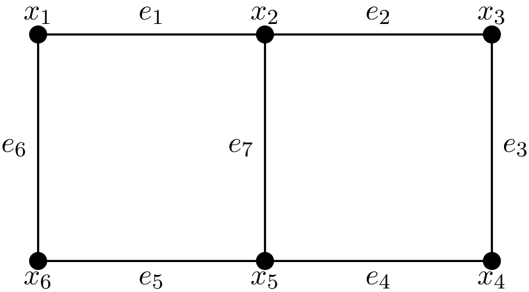
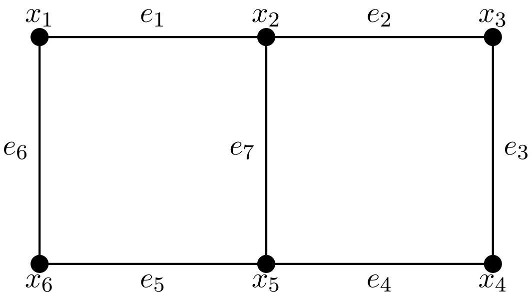
Theorem (Villarreal)

$I_G = \langle f_W \mid W \text{ an even closed walk in } G \rangle$.

Example

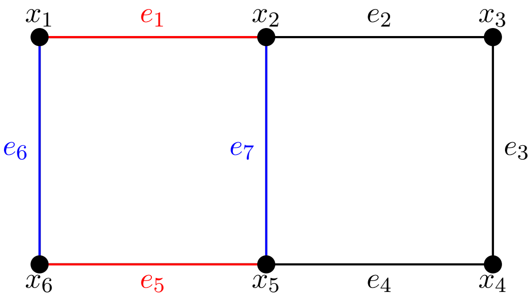


Example



$$I_G = \langle e_1e_5 - e_6e_7, e_2e_4 - e_3e_7, e_1e_3e_5 - e_2e_4e_6 \rangle$$

Example



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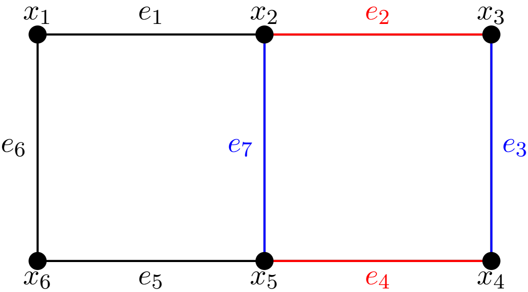

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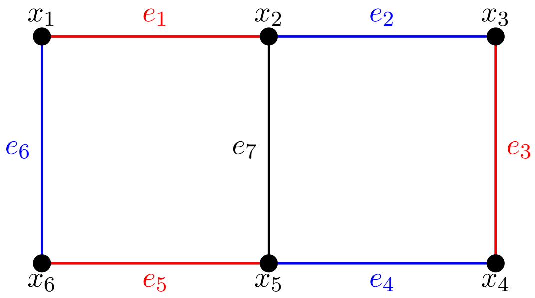

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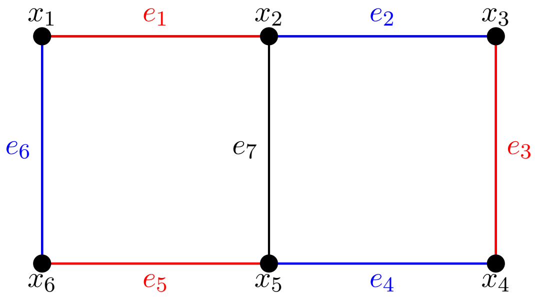

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$$e_1e_3e_5 - e_2e_4e_6 = e_3(e_1e_5 - e_6e_7) - e_6(e_2e_4 - e_3e_7)$$

Weighted Oriented Graphs

A (vertex)-weighted oriented graph $\mathcal{D}$ is a triple $\mathcal{D} = (V, E, \mathbf{w})$ where

- $V = \{x_1, \ldots, x_n\}$ is the vertex set
- E is a set of ordered pairs of elements of V where (x_i, x_j) represents an edge from x_i to x_j
- $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{N}^n$ is a vector which assigns weights to the vertices of $\mathcal{D}$

Toric Ideals of weighted oriented graphs

Given a weighted oriented graph $\mathcal{D} = (V, E, w)$ on vertex set $V = \{x_1, \ldots, x_n\}$ and edge set $E = \{e_1, \ldots, e_t\}$ we define a map

$$\begin{aligned} \varphi : k[e_1, \ldots, e_t] &\to k[x_1, \ldots, x_n] \\ e_i &\mapsto x_{i_1} x_{i_2}^{w_{i_2}} \end{aligned}$$

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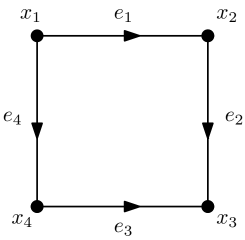
The kernel of φ is the *toric ideal of the weighted oriented graph* $\mathcal{D}$ which we denote $I_{\mathcal{D}}$.

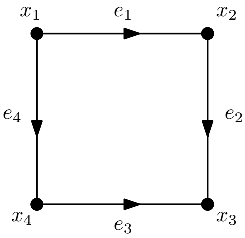
Examples



$\mathbf{w} = (2, 3, 4, 3)$

$$e_1 \mapsto x_1 x_2^3 \qquad e_2 \mapsto x_2 x_3^4$$
$$e_3 \mapsto x_4 x_3^4 \qquad e_4 \mapsto x_1 x_4^3$$

Examples



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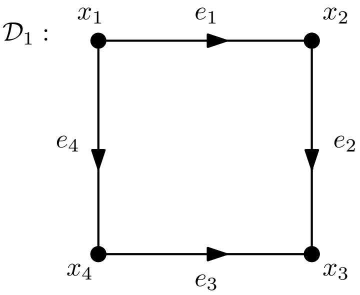
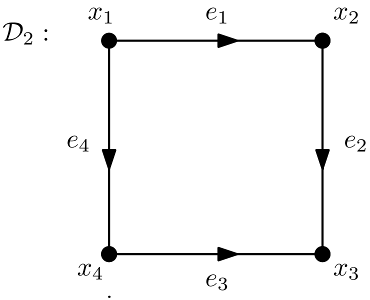
$$I_{\mathcal{D}} = \langle e_1 e_3^3 - e_2^3 e_4 \rangle$$



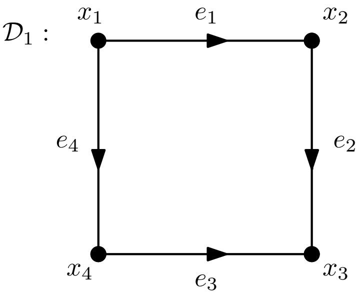
$\mathcal{D}_1$:
x_1
e_1
x_2
e_4
e_2
x_4
e_3
x_3

$(2, 3, 4, 3)$

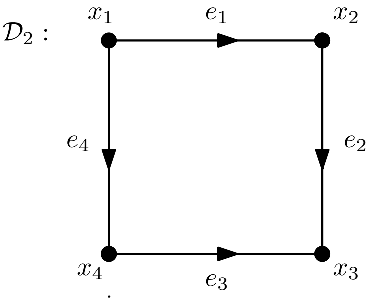


$\mathcal{D}_2$:
x_1
e_1
x_2
e_4
e_2
x_4
e_3
x_3

$(2, 3, 3, 4)$



$(2, 3, 4, 3)$



$(2, 3, 3, 4)$

$$I_{\mathcal{D}_1} = \langle e_1 e_3^3 - e_2^3 e_4 \rangle$$
$$I_{\mathcal{D}_2} = \{0\}$$

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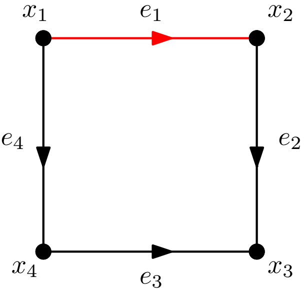
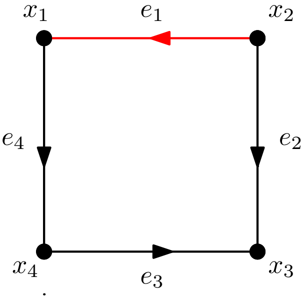
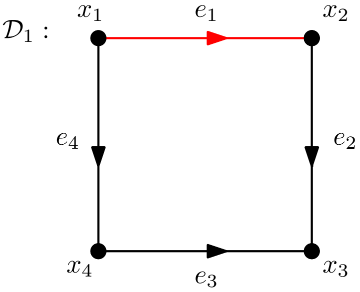


$(2, 3, 4, 3)$

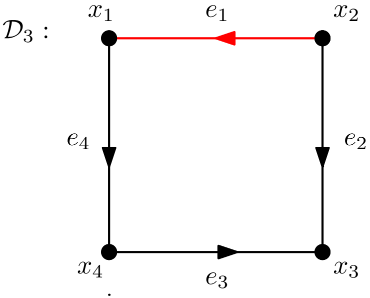
$\mathcal{D}_3$:



$(2, 3, 4, 3)$



$(2,3,4,3)$



$(2,3,4,3)$

$$I_{\mathcal{D}_1} = \langle e_1 e_3^3 - e_2^3 e_4 \rangle$$
$$I_{\mathcal{D}_3} = \{0\}$$

If $\mathcal{C}_n$ is a weighted oriented cycle on n vertices, we say $\mathcal{C}_n$ is *balanced* if n is even and the products of the weights on the heads of the arrows pointing counter-clockwise is equal to the products of the weights on the heads of the arrows pointing clockwise.

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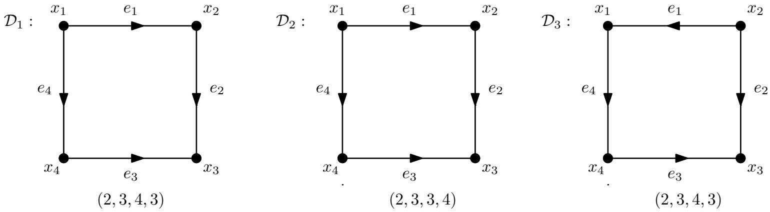
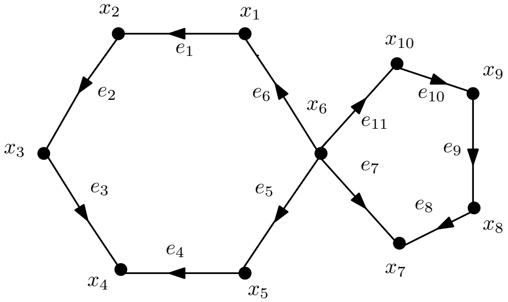


$\mathcal{D}_1$ is balanced while $\mathcal{D}_2$ and $\mathcal{D}_3$ are unbalanced.

Theorem (B-Kara-Lin-O'Keefe)

Let $\mathcal{D}$ be a weighted oriented graph with a single cycle. Then the toric ideal $I_{\mathcal{D}}$ is non-zero if and only if the cycle is balanced.

Two Cycles - Example 1



$\mathbf{w} = (2, 2, 3, 4, 6, 1, 4, 2, 3, 2)$

The toric ideal of the underlying graph is $\langle e_1e_3e_5 - e_2e_4e_6 \rangle$.

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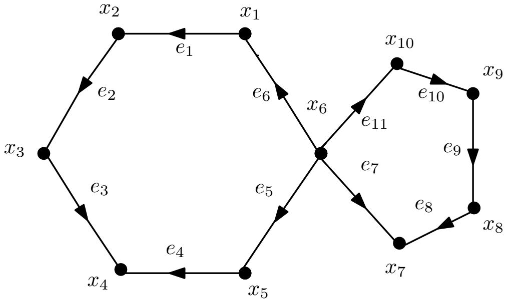

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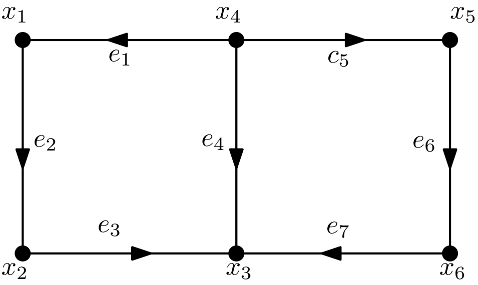
The toric ideal of the underlying graph is $\langle e_1 e_3 e_5 - e_2 e_4 e_6 \rangle$.

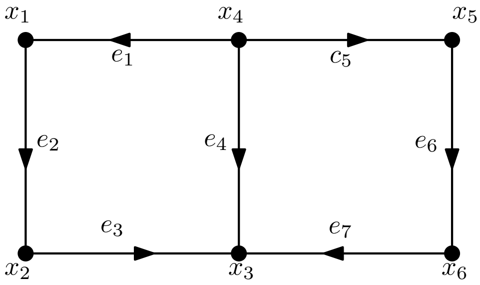
Here, $I_{\mathcal{D}} = \langle e_2^{52} e_4^{156} e_6^{13} e_7^{12} e_9^{6} e_{11} - e_1^{26} e_3^{156} e_5^{26} e_8^{12} e_{10}^{2} \rangle$.



$$\mathbf{w} = (2, 2, 3, 1, 2, 2)$$

The toric ideal of the underlying graph is $\langle e_1 e_3 - e_2 e_4, e_4 e_6 - e_5 e_7 \rangle$.

Two Cycles - Example 2



$\mathbf{w} = (2, 2, 3, 1, 2, 2)$

The toric ideal of the underlying graph is $\langle e_1e_3 - e_2e_4, e_4e_6 - e_5e_7\rangle$.

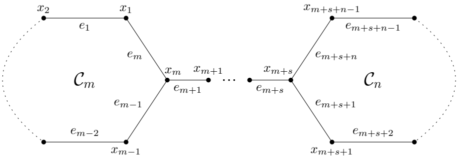
Here, $I_{\mathcal{D}} = \langle e_1e_3^4e_6^2 - e_2^2e_5e_7^4\rangle$.

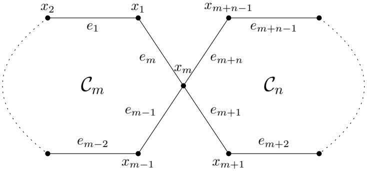
Theorem (B-Kara-Lin-O'Keefe)

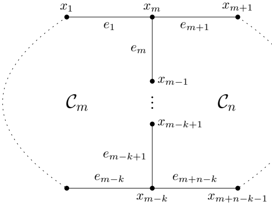
Let $\mathcal{D}$ be a weighted oriented graph comprised of two oriented cycles $\mathcal{C}_m$ and $\mathcal{C}_n$ satisfying one of the following:

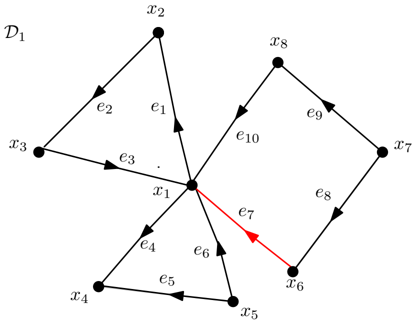
- $\mathcal{C}_m$ and $\mathcal{C}_n$ share a single vertex,
- the set $E(\mathcal{C}_m) \cap E(\mathcal{C}_n)$ induces an oriented path with at least one edge, or
- $\mathcal{C}_m$ and $\mathcal{C}_n$ are connected by an oriented path $\mathcal{P}_s$ of length $s \geq 1$.

Then the toric ideal $I_{\mathcal{D}}$ is generated by a single element if and only if at most one of its cycles is balanced.

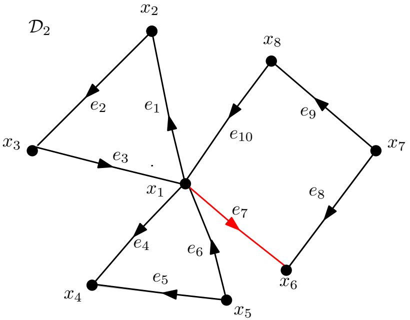

x_2
x_1
e_1
e_m
x_m
x_{m+1}
$\cdots$
x_{m+s}
e_{m+1}
e_{m+s}
$\mathcal{C}_m$
e_{m-1}
e_{m-2}
x_{m-1}
$x_{m+s+n-1}$
$e_{m+s+n-1}$
e_{m+s+n}
$\mathcal{C}_n$
e_{m+s+1}
e_{m+s+2}
x_{m+s+1}


x_2
x_1
e_1
x_{m+n-1}
e_{m+n-1}
e_m
x_m
e_{m+n}
$\mathcal{C}_m$
$\mathcal{C}_n$
e_{m-1}
e_{m+1}
e_{m-2}
e_{m+2}
x_{m-1}
x_{m+1}


x_1
x_m
x_{m+1}
e_1
e_{m+1}
e_m
x_{m-1}
$\mathcal{C}_m$
$\vdots$
$\mathcal{C}_n$
x_{m-k+1}
e_{m-k+1}
e_{m-k}
e_{m+n-k}
x_{m-k}
$x_{m+n-k-1}$



$\mathbf{w} = (2, 2, 2, 2, 1, 2, 1, 2)$



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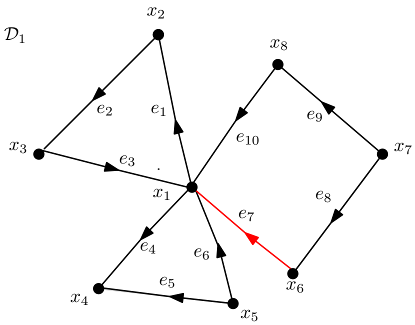

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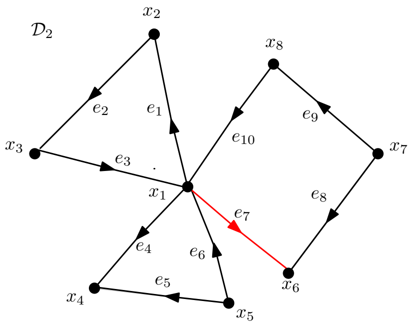
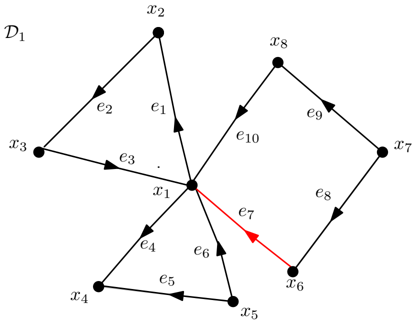


$\mathbf{w} = (2, 2, 2, 2, 1, 2, 1, 2)$

$$I_{\mathcal{D}_2} = \langle e_7^2 e_9 - e_8 e_{10}^2, e_1 e_3^4 e_5^3 - e_2^2 e_4^3 e_6^3 \rangle$$



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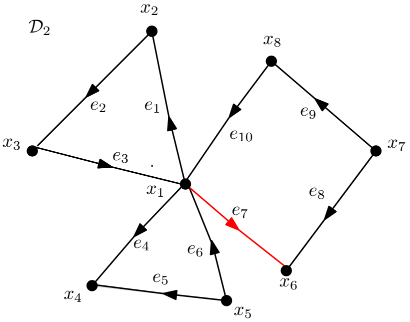

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$$I_{\mathcal{D}_2} = \left\langle \begin{array}{ll} e_4 e_6 e_7 e_9 - e_5 e_8 e_{10}^2, & e_1 e_3^4 e_5^3 - e_2^2 e_4^3 e_6^3, \\ e_1 e_3^4 e_5^2 e_7 e_9 - e_2^2 e_4^2 e_6^2 e_8 e_{10}^2, & e_1 e_3^4 e_5 e_7^2 e_9^2 - e_2^2 e_4 e_6 e_8^2 e_{10}^4, \\ e_1 e_3^4 e_7^3 e_9^3 - e_2^2 e_8^3 e_{10}^6 & \end{array} \right\rangle$$