Interpolation problem and Chudnovsky's Conjecture

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Interpolation problem

Question 1.1 (Interpolation Problem)

What is the least degree $\alpha_{\overline{m}}(\mathbb{X})$ *of a homogeneous polynomial that vanishes at the points* $\mathbb{X} = \{P_1, \dots, P_s\}$ *in* $\mathbb{P}^N_{\mathbb{C}}$ *with multiplicities at least* $\overline{m} = (m_1, \dots, m_s)$ *respectively ?*

Equi-multiplicity: $m_1 = \cdots = m_s = m$

Goal is to study lower bounds of

$$\alpha_m(\mathbb{X}) = \text{least degree of a homogeneous polynomial that vanishes at } \mathbb{X} \text{ at least } m \text{ times}$$

Conjectural Lowers bounds on $\alpha_m(\mathbb{X})$

- $\mathbb{X} = \{P_1, \ldots, P_s\} \subset \mathbb{P}^N_{\mathbb{C}}$.
- Nagata'65: $\dfrac{\alpha_m(\mathbb{X})}{m} \geqslant \sqrt{s}$, for at least 10 general points $\mathbb{P}^2_{\mathbb{C}}$
- Chudnovsky'81: $\dfrac{\alpha_m(\mathbb{X})}{m} \geqslant \dfrac{\alpha_1(\mathbb{X}) + N - 1}{N}$, $\forall m \geqslant 1$. Points in $\mathbb{P}^N_{\mathbb{C}}$
- Demailly'82: $\dfrac{\alpha_m(\mathbb{X})}{m} \geqslant \dfrac{\alpha_t(\mathbb{X}) + N - 1}{t + N - 1}$, $\forall m, t \geqslant 1$ Points in $\mathbb{P}^N_{\mathbb{C}}$

Symbolic powers and Geometry

- $I \subset R = \mathbb{K}[x_0, \ldots, x_N]$ homogeneous;
- Symbolic Powers: $I^{(m)} = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(R/I)} (I^m R_{\mathfrak{p}} \cap R)$.
- If $I_{\mathbb{X}} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_s$ is the defining ideal of $\mathbb{X}$, $\mathfrak{p}_i$ is the defining ideal of the point P_i, then

$$\begin{aligned} I_{\mathbb{X}}^{(m)} &= \mathfrak{p}_1^m \cap \cdots \cap \mathfrak{p}_s^m, \\ &= \text{Polynomials vanishing at } \mathbb{X} \text{ at least } m \text{ times }. \end{aligned}$$

- Goal is to study bounds on the least degree element in $I_{\mathbb{X}}^{(m)}$.

Initial Degree and the Waldschmidt Constant

- The initial degree: $\alpha(I) =$ the least degree of an element in I.
- The sequence $\{\alpha(I^{(t)})\}_{t\in\mathbb{N}}$ is sub-additive for any ideal I. Hence, by Fekete's lemma,

$$\widehat{\alpha}(I) = \lim_{t\to\infty} \frac{\alpha(I^{(t)})}{t} = \inf_{t\in\mathbb{N}} \frac{\alpha(I^{(t)})}{t} \text{ is well defined.}$$

- $\widehat{\alpha}(I)$ is known as the Waldschmidt constant of I.
- Goal: Study bounds for $\alpha\left(I_{\mathbb{X}}^{(m)}\right)$ or $\widehat{\alpha}(I_{\mathbb{X}})$, where $\mathbb{X} = \{P_1, \dots, P_s\}$.

Equivalent Statements of the Conjectures

- $I_{\mathbb{X}}$ defining ideal of $\mathbb{X} = \{P_1, \ldots, P_s\} \subset \mathbb{P}^N_{\mathbb{C}}$.
- Nagata'65: $\widehat{\alpha}(I_{\mathbb{X}}) \geqslant \sqrt{s}$, for at least 10 very general points $\mathbb{P}^2_{\mathbb{C}}$.
- Chudnovsky'81: $\widehat{\alpha}(I_{\mathbb{X}}) \geqslant \dfrac{\alpha(I_{\mathbb{X}}) + N - 1}{N}$, Points in $\mathbb{P}^N_{\mathbb{C}}$.
- Demailly'82: $\widehat{\alpha}(I_{\mathbb{X}}) \geqslant \dfrac{\alpha(I^{(t)}_{\mathbb{X}}) + N - 1}{t + N - 1}$, $\forall t \geqslant 1$ Points in $\mathbb{P}^N_{\mathbb{C}}$.

Results: Chudnovsky's Conjecture

- General points in $\mathbb{P}^2_{\mathbb{C}}$ (Chudnovsky'81; Harbourne - Huneke'13).
- General points in $\mathbb{P}^3_{\mathbb{K}}$ ($char \mathbb{K} = 0$) (Dumnicki'12).
- $\leqslant N+1$ general points in $\mathbb{P}^N_{\mathbb{K}}$ ($char \mathbb{K} = 0$) (Dumnicki'12).
- Binomial number of points in $\mathbb{P}^N_{\mathbb{K}}$ forming a star configuration (Bocci - Harbourne '10).
- 15 linearly general points in $\mathbb{P}^4_{\mathbb{K}}$ (Thomas'21).

Results: Chudnovsky's Conjecture

- $\geqslant 2^N$ very general points in $\mathbb{P}^N_{\mathbb{K}}$ (Dumnicki - Tutaj-Gasińska'16).
- Very general points in $\mathbb{P}^N_{\mathbb{K}}$; also $\leqslant \binom{N+2}{2} - 1$ points (Fouli - Mantero - Xie'16).
- $\leqslant N+4$ points in $\mathbb{P}^N_{\mathbb{K}}$ (Nagel - Trok'19).
- $\geqslant 3^N (N \geqslant 4$, reduces to 2^N for $N \geqslant 9)$ general points in $\mathbb{P}^N_{\mathbb{K}}$ (B - Grifo - Hà - Nguyễn'20).
- Remaining cases of general points in $\mathbb{P}^N_{\mathbb{K}}$ for $N \geqslant 4$; (B - Nguyễn'22) (Focus of this talk).

General and very general set of points

- A property $\mathcal{P}$ holds for Very General set of s points when
  - property $\mathcal{P}$ holds for points obtained from an infinite intersection of open subsets in an appropriate affine space, via coordinatewise specialization.
- A property $\mathcal{P}$ holds for General set of s points when
  - property $\mathcal{P}$ holds for points obtained from one open subset in an appropriate affine space, via coordinatewise specialization.

–by Fouli - Mantero - Xie'16

Harbourne-Huneke Conjecture

There are some pioneer containment results. We focus on the following.

Conjecture 2.1 (Harbourne - Huneke'13)

If $R = \mathbb{K}[\mathbb{P}^N_{\mathbb{K}}]$, $I \subset R$, *homogeneous radical with bigheight* $= h$, *and* $\mathfrak{m} = \langle x_0, \dots, x_N \rangle$ *then*

$$I^{(hr)} \subseteq \mathfrak{m}^{r(h-1)} I^r, \text{ for all } r \geqslant 1.$$

This containment implies Chudnovsky's Conjecture

Conjecture 2.2 (Stable Harbourne-Huneke Containment)

If $R = \mathbb{K}[\mathbb{P}^N_{\mathbb{K}}]$, $I \subset R$, *homogeneous radical bigheight* $= h, \mathfrak{m} = \langle x_0, \dots, x_N \rangle$, *then*

$$I^{(hr)} \subseteq \mathfrak{m}^{r(h-1)} I^r, \text{ for } r \gg 0.$$

Cremona Transformation

Definition 2.1

The standard birational transformation

$$\Phi : \mathbb{P}^N \to \mathbb{P}^N \text{ defined by } \Phi(x_0 : \cdots : x_N) \dashrightarrow (x_0^{-1} : \cdots : x_N^{-1}),$$

is known as Cremona transformation.

Why we used Cremona transformation? To reduce accordingly.

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Theorem 2.2 (B-Nguyễn'22 (inspired by Dumniscki's results of $N = 3$))

Let $I(m_1, \dots, m_s) \equiv s$ *generic pts with multiplicities* $m_1, \dots, m_s$ *resp and* $k = (N-1)d - \sum_{j=1}^{N+1} m_j, N \geqslant 4$. *If* $I(m_1, \dots, m_s)_d \neq 0$, *then*

$$I(m_1 + k, \dots, m_{N+1} + k, m_{N+2}, \dots, m_s)_{d+k} \neq 0.$$

Example: 8 generic points in $\mathbb{P}^4$

Example 2.3 (B-Nguyễn'22)

d	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	k
7	$\underline{5}$	$\underline{5}$	$\underline{5}$	$\underline{5}$	$\underline{5}$	5	5	5	-4
3	1	1	1	1	1	5	5	5	$*$

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- Suppose that $I\left(5^{\times 8}\right)_7 = \left[I^{(5)}\right]_7 \neq 0$.

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- The reduction table shows $I\left(5^{\times 8}\right)_7 \neq 0 \implies I\left(1^{\times 5}, 5^{\times 3}\right)_3 \neq 0 \Rightarrow\Leftarrow$.

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- Suppose that $I\left(5^{\times 8}\right)_7 = \left[I^{(5)}\right]_7 \neq 0$.
- The reduction table shows $I\left(5^{\times 8}\right)_7 \neq 0 \implies I\left(1^{\times 5}, 5^{\times 3}\right)_3 \neq 0 \Rightarrow\Leftarrow$.
- Similarly, $I\left((5m)^{\times 8}\right)_{8m-1} = 0, \forall m \geqslant 1$, which implies,

$$\widehat{\alpha}(I(8)) \geqslant \frac{8}{5}.$$

Reduction Inspired by Dumniscki's Results for $N = 3$

Theorem 2.4 (B-Nguyễn'22)

If $N \geqslant 4$, $I(m^{\times c}) = I(\underbrace{m, \ldots, m}_{c \text{ many}})$, *and* $\widehat{\alpha}(s) = \widehat{\alpha}(I(1^{\times s}))$ *(generic), then*

1. $\widehat{\alpha}(b \cdot 2^N) \geqslant 2\widehat{\alpha}(b)$;
2. $\widehat{\alpha}(I(1^{\times b \cdot 2^N}, \overline{m})) \geqslant \widehat{\alpha}(I(2^{\times b}, \overline{m}))$.

Example 2.5

Consider 128 generic points in $\mathbb{P}^4$. Then

$$\widehat{\alpha}(128) = \widehat{\alpha}\,(\underbrace{8 \cdot 16}_{8 \times 2^4}) \geqslant 2\widehat{\alpha}(8) \geqslant 2 \cdot \frac{8}{5} = \frac{16}{5}.$$

Again, $3^4 \leqslant 128 \leqslant 4^4$, which implies that $\widehat{\alpha}(128) \geqslant 3$.

Reduction of Dimension

Theorem 2.6 (Dumnicki-Szemberg-Spondz'18)

For appropriate choices of r_j, and a_j (condition explained in their article), if

$$\widehat{\alpha}(\mathbb{P}^{N-1}, r_j) \geqslant a_j, \text{ for all } j = 1, \ldots, k+1, \text{(very general)}$$

then

$$\widehat{\alpha}(\mathbb{P}^{N}, r_1 + \ldots + r_{k+1}) \geqslant \left(1 - \sum_{j=1}^{k} \frac{1}{a_j}\right) a_{k+1} + k. \text{(very general)}$$

Three reductions:

- Reduction of multiplicities
- Reduction of number of points
- Reduction of dimensions

Chudnovsky's Conjecture: Results

Theorem 2.7 (B-Nguyễn '22)

Remaining cases of general points in $\mathbb{P}^N$ for $N \geqslant 4$ satisfies the following:

- *Stable Harbourne-Huneke:* $I^{(hr)} \subseteq \mathfrak{m}^{r(h-1)} I^r$, for $r \gg 0$.
- *Chudnovsky's Conjecture:* $\widehat{\alpha}(I) \geqslant \dfrac{\alpha(I) + N - 1}{N}$.

Thank you :)

Figure: CAAC 23