Snake Graphs for Graph LP Algebras

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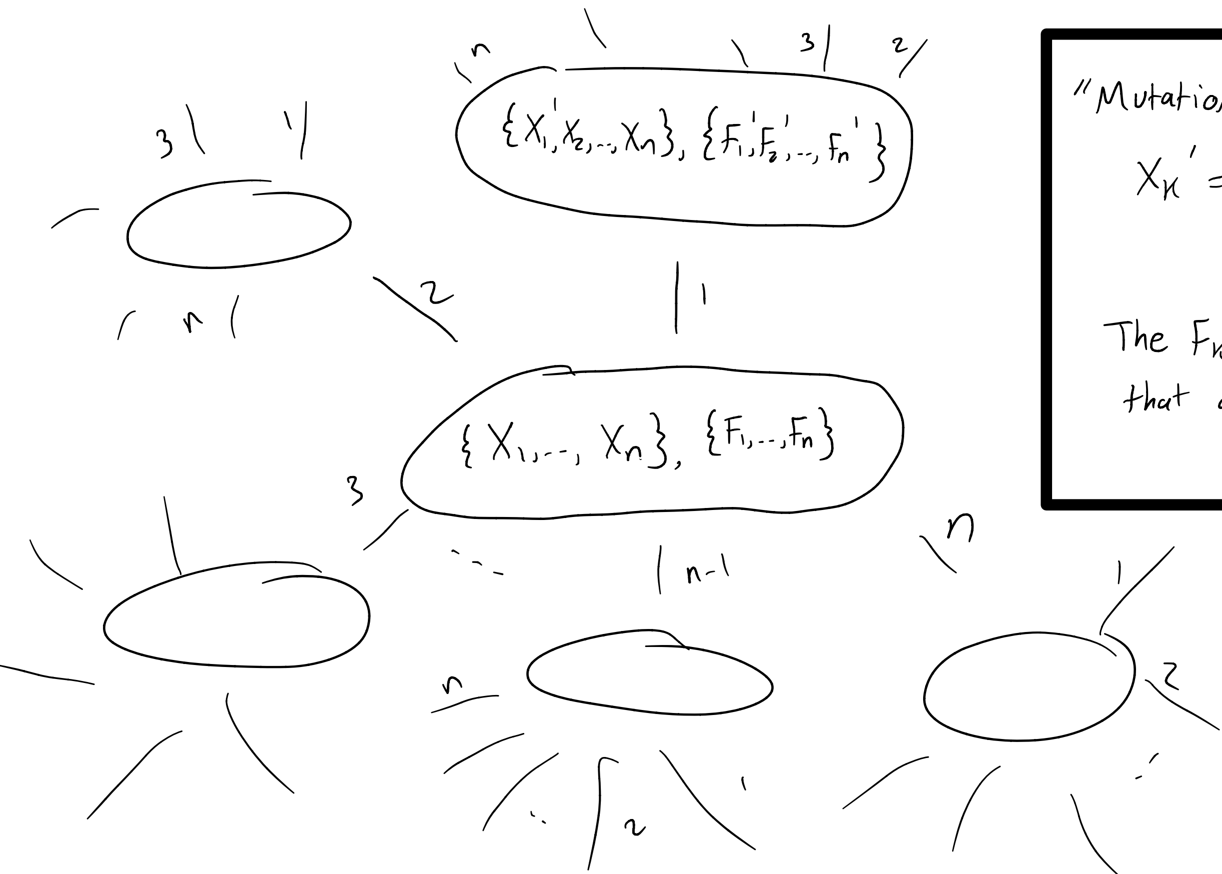
Laurent Phenomenon (LP) Algebras

Lam-Pylyavskyy 2012



"Mutation in direction k"

$$X_k' = \frac{F_k(X_1, X_2, \ldots, X_{k-1}, X_{k+1}, \ldots X_n)}{X_k}$$

The F_k satisfy properties guaranteeing that all variables are <u>Laurent polynomials</u>

(i.e. denominator is monomial)

Graph LP Algebras

Lam-Pylyavskyy 2012

Let Γ be an undirected graph. We will only consider trees.

Variables: $\{X_i \mid i \in V(\Gamma)\} \cup \{Y_S \mid S \subseteq V(\Gamma),\ S \text{ is connected}\}$

Clusters: $\{Y_I \mid I \in \mathcal{I}$ is a maximal nested collection on $W \subseteq V(\Gamma)\} \cup \{X_i \mid i \in V(\Gamma) \setminus W\}$

We work with $W = V(\Gamma)$.

maximal set of compatible variables

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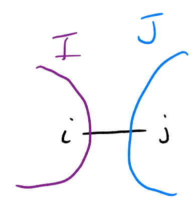
maximal set of compatible variables

Def: A set $\mathcal{I}$ of subsets of $V(\Gamma)$ is a nested collection if for all $I, J \in \mathcal{I}$, either

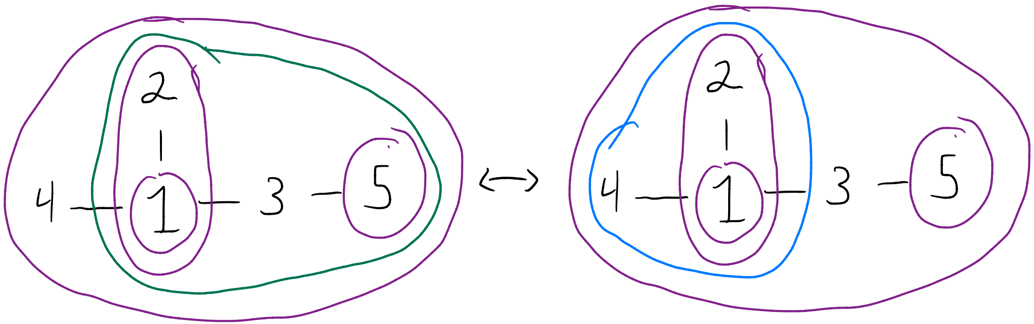
(1) $I \subseteq J$ or $J \subseteq I$, or

(2) $I \cap J = \emptyset$ and I & J do not "kiss"



Graph LP Algebras: mutation

Example of Mutation:



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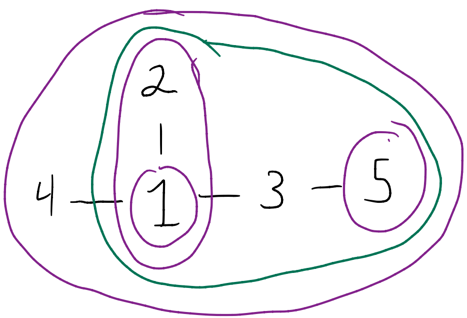
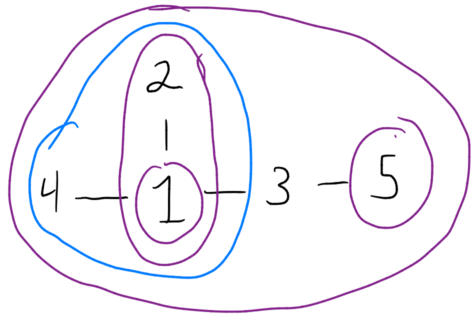


$\longleftrightarrow$



$\{1235\} \cup \{124\}$

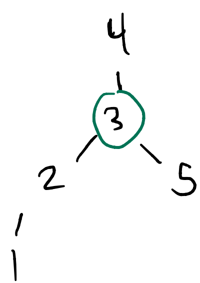
$\{1235\} \cap \{124\}$

$$Y_{1235}\, Y_{124} = Y_{12345}\, Y_{12} + Y_5\, Y_2^{\,2}$$

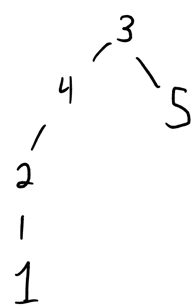
$\{1235\} \setminus \{124\}$

$\{123\} \cap \{124\} \setminus \text{path } 3-4$

$\mathcal{I} \to$



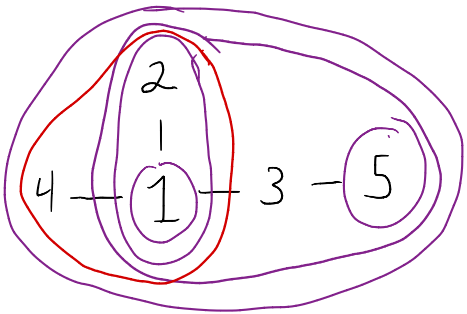
$\mathcal{I}' \to$



Difficulty #1)

$\{2\} \notin \mathcal{I}, \mathcal{I}'$

Graph LP Algebras: mutation

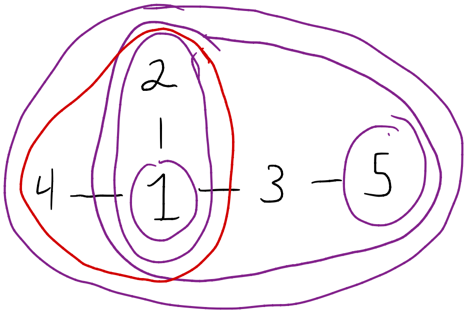


Substituting $Y_2 = \dfrac{Y_{12}+1}{Y_1}$ we can write Y_{124} in terms of $\mathcal{I}$,

$$Y_{124}^{\mathcal{I}} = \frac{Y_{12345}\, Y_{12}\, Y_1^{2} + Y_{12}^{2} Y_5 + 2\, Y_{12}\, Y_5 + Y_5}{Y_{1235}\, Y_1^{2}}$$

Difficulty #2: Y_1 appears in the denominator, but $\{1\}$ and $\{1,2,4\}$ are compatible – $\{1\} \subset \{1,2,4\}$

Graph LP Algebras: mutation



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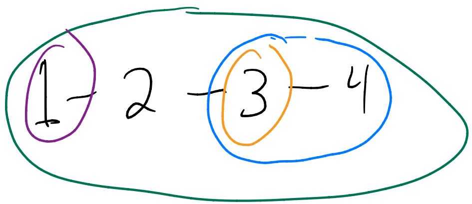
Goal: Prove $Y_S^{\mathcal{I}}$ have positive coefficients

Special Case: Path Graphs

There is a bijection between maximal nested collections on P_n and triangulations of an $(n+2)$-gon:



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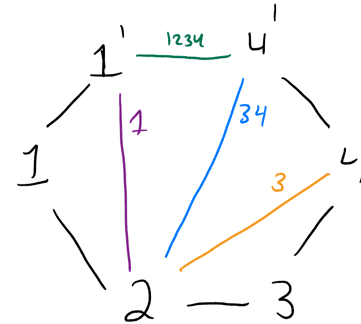

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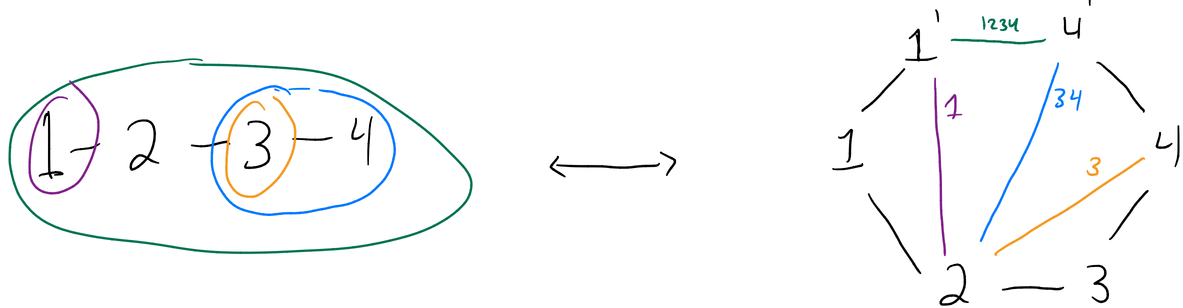
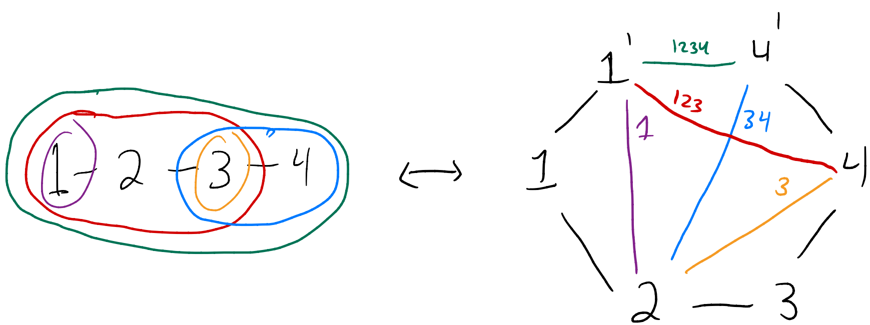


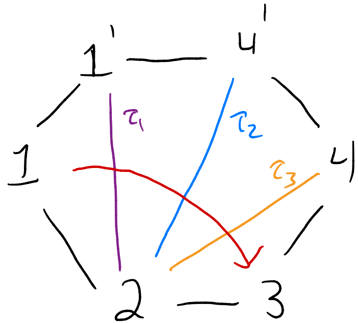
Here, mutation of Y-variables resembles mutation in a cluster algebra of type A.



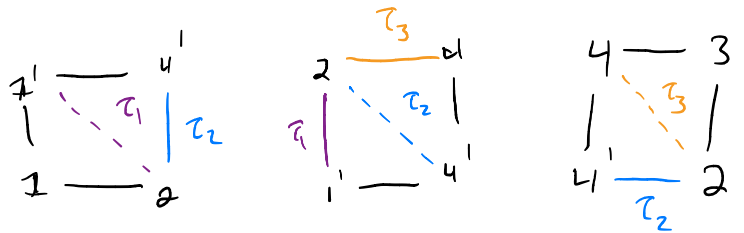
$$Y_{123}\, Y_{34} = Y_{1234}\, Y_3 + Y_1$$

Snake Graphs

Musiker - Schiffler



- one tile for each crossing of γ with an arc in $T = \{\tau_1, \tau_2, \tau_3\}$



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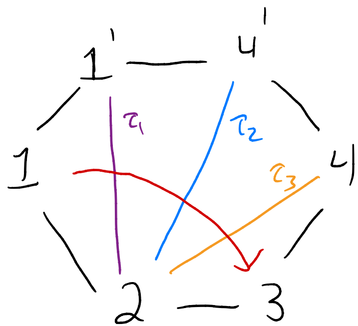

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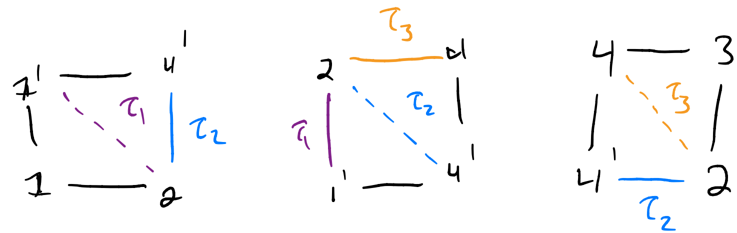
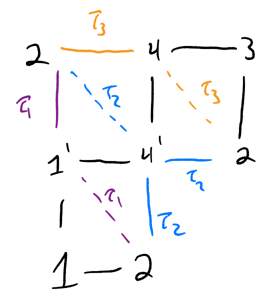


- glue tiles associated to adjacent crossings

$G_{\gamma,T}$



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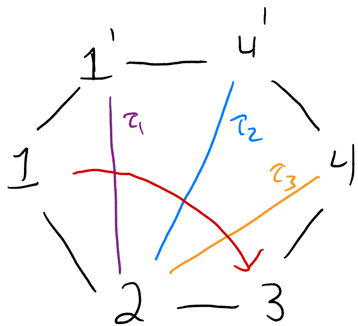

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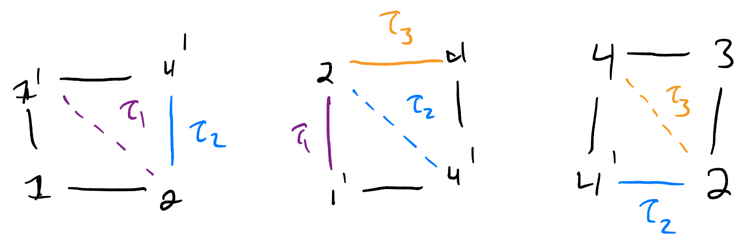

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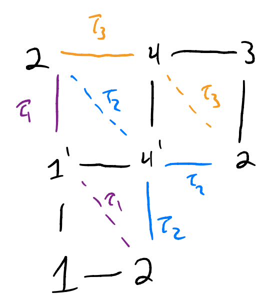
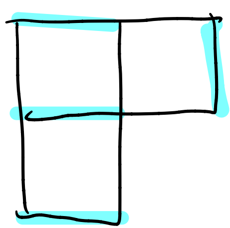




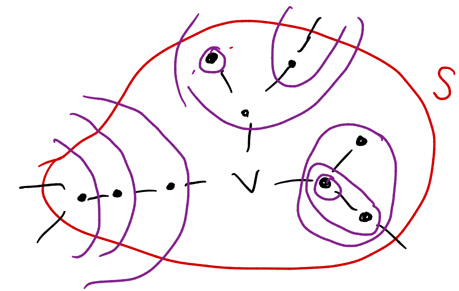
- The cluster variable associated to γ is given by statistics from the set of perfect matchings of $\mathcal{G}_{\gamma,T}$

such as

Main Result

Def A set S is <u>weakly rooted</u> with respect to a maximal nested collection $\mathcal{I}$ if there exists a root v so that, for any $i, j \in S$, either

- "$I_i \subseteq I_j$ iff j appears on the path $i - v$" OR
- $\exists\, I \in \mathcal{I}$ such that $i, j \in I \subseteq S$.



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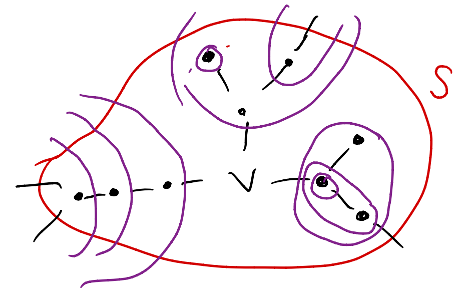


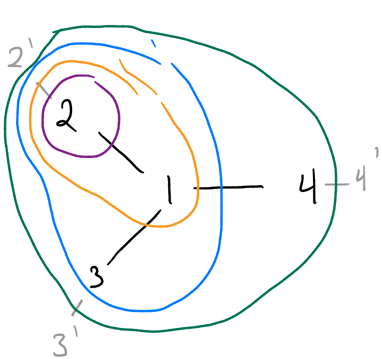
Thm [B.-Chepuri-Kelley-Zhang]
For any maximal nested collection $\mathcal{I}$ and weakly rooted set S,

$$Y_S = \frac{1}{\text{labels}(\mathcal{G}_S)} \sum_{P} wt(P)$$

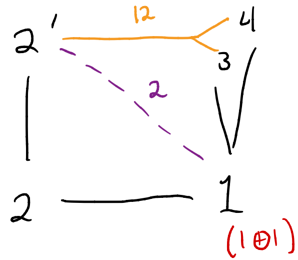
"admissible matchings of $\mathcal{G}_S$"

Cor For some $(S, \mathcal{I})$, this shows Y_S has positive coefficients

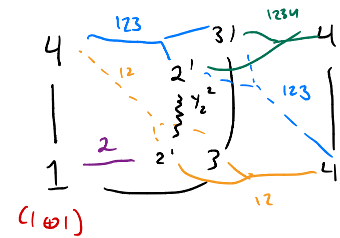
Examples



$\mathcal{G}_1 =$



$\mathcal{G}_4 =$



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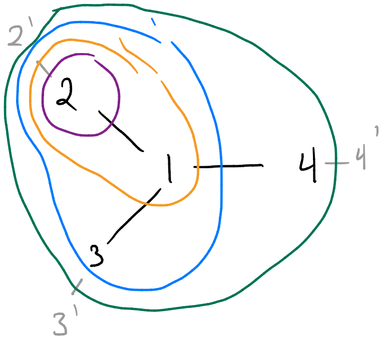

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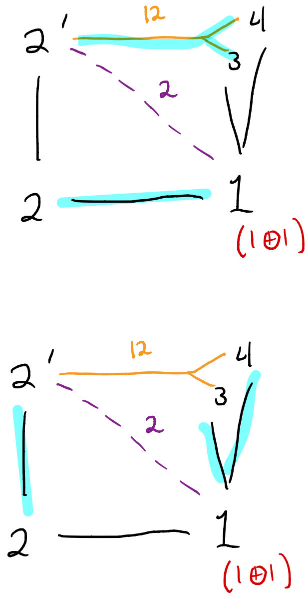
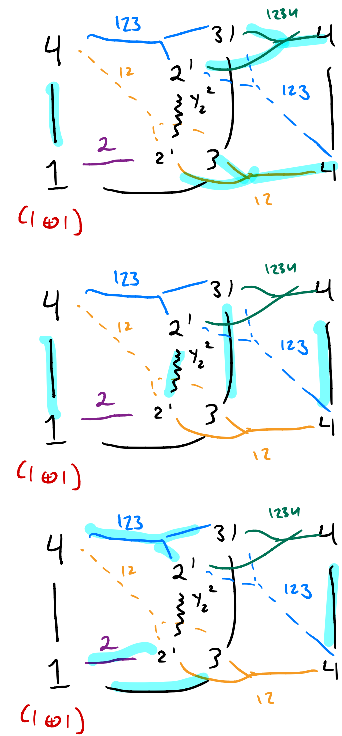


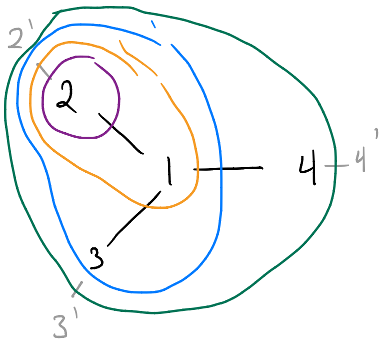
$\mathcal{G}_4 =$



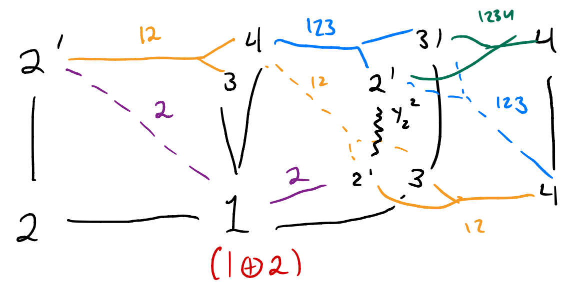
$$Y_1 = \frac{Y_{12} + 1}{Y_2}$$

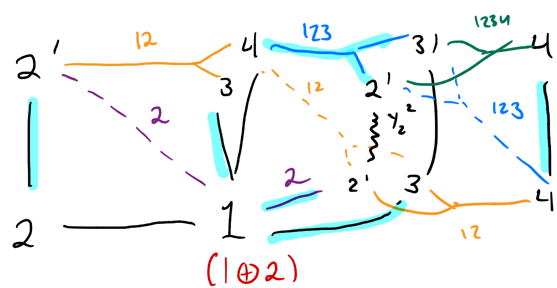
$$Y_4 = \frac{Y_{1234} Y_2 + Y_2^2 + Y_{123} Y_2}{Y_{123} Y_{12}}$$

Examples



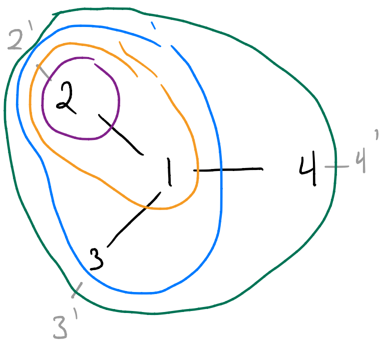
$G_{14} =$

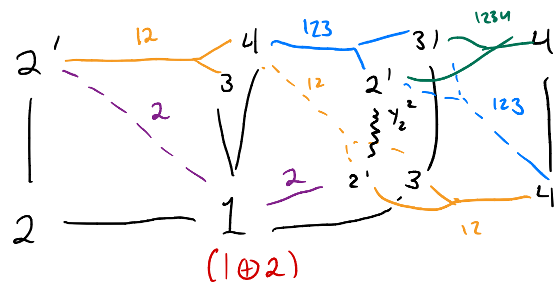


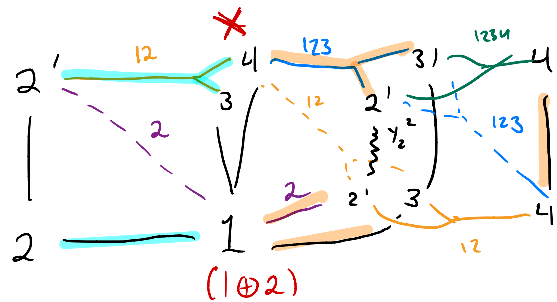


Contributes $\dfrac{Y_2 \, Y_{123}}{Y_2 \, Y_{12} \, Y_{123}}$

Examples



$\mathcal{G}_{14} =$ 



would contribute $\dfrac{Y_{12} \cdot Y_2 Y_{123}}{Y_2 \cdot Y_{12} Y_{123}}$

Relation: $Y_{14} = Y_1 Y_4 - 1$

Graphically, we can combine a matching of $\mathcal{G}_1$ & a matching of $\mathcal{G}_4$ except in one case.

Thank you

for listening!