Unipotent Wilf Conjecture

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Wilf Conjecture

Let S be a complement finite submonoid of $\mathbb{N}_0$, (a.k.a numerical semigroup).

- The **conductor** of S, denoted by $c(S)$ is the smallest integer c such that $c + \mathbb{N} \subseteq S$.
- Let $n(S)$ denote the cardinality of the set
$$\{x \in S : x < c\}.$$
- Let $e(S)$ denote the cardinality of the minimal generating set of $S \backslash \{0\}$.

In 1978, Wilf conjectured that [1] for any numerical semigroup $S \subseteq \mathbb{N}_0$, we have
$$c(S) \leq e(S)n(S)$$

Example: Let $S = \{0, 3, 5, 6, 8 \dots\}$. Then $c(S) = 7$, $n(S) = 4$ and $e(S) = 2$.

Previous Generalization

Let S be a complement finite submonoid of $\mathbb{N}_0^d$ (a.k.a generalized numerical semigroup).

Let $\leq$ be a partial order on $\mathbb{N}_0^d$ such that for $x = (x_1, \dots, x_d)$, $y = (y_1, \dots, y_d) \in \mathbb{N}_0^d$, $x \leq y$ if and only if $x_i \leq y_i$ for all $i = 1, \dots, r$. Let $H(S) = \mathbb{N}_0^d \backslash S$. We define

- The **conductor** of S, denoted by $\mathrm{c}(S)$ is the cardinality of the set
$$\{x \in \mathbb{N}_0^d \ : \ x \leq h \text{ for some } h \in H(S)\}$$
- Let $\mathrm{n}(S)$ denote the cardinality of the set
$$\{x \in S \ : \ x \leq h \text{ for some } h \in H(S)\}$$
- Let $\mathrm{e}(S)$ denote the cardinality of the minimal set of generators of S.

Generalized Wilf Conjecture [2] states that
$$d\mathrm{c}(S) \leq \mathrm{e}(S)\mathrm{n}(S)$$

Important families

Let G be a unipotent complex linear algebraic group. It is well known that G is isomorphic to a closed subgroup of the unipotent upper triangular $n \times n$ matrices with entries in $\mathbb{C}$, denoted as $\mathbf{U}(n, \mathbb{C})$. Define
$$\mathbf{U}(n, \mathbb{N}_0)_k = \{(x_{ij}) : k \leq \max_{1 \leq i < j \leq n} \{x_{ij}\}\}.$$

The commutative subgroup lives in
$$\mathbf{P}(n, \mathbb{N}_0) := \left\{ \begin{pmatrix} 1 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} : a_i \in \mathbb{N}_0 \right\} \cong \mathbb{N}_0^{n-1}$$

Our Notations

Let G be a unipotent complex linear algebraic group and let $M = G_{\mathbb{N}} \subseteq \mathbf{U}(n, \mathbb{N}_0)$. Let $S \subseteq M$ be complement finite submonoid.

- The generating number of S is defined as $\mathrm{r}_M(S) = \min\{k \in \mathbb{N} : \mathbf{U}(n, \mathbb{N}_0)_k \cap M \subseteq S\}$.
- $\mathbf{d}_M := \dim G$.
- $\mathbf{c}_M(S) := \mathrm{r}(S)^{\mathbf{d}_M}$. *(Conductor of S.)*
- $\mathbf{n}_M(S) := |S \backslash \mathbf{U}(n, \mathbb{N})_{\mathbf{r}_M(S)}| + 1$.
- $\mathbf{e}(S) := \min\{|\mathscr{G}| \ : \ \mathscr{G} \text{ generates } S \backslash \{\mathbf{1}_n\}\}$.
- $\mathbf{g}(S) := |M \backslash S|$. *(Genus of S relative to M.)*

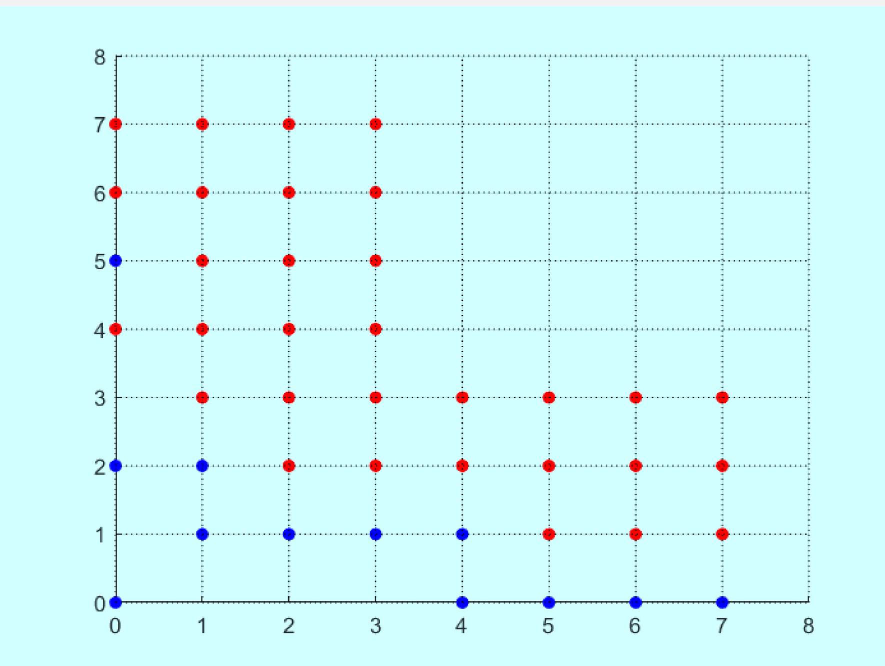
Unipotent Wilf Conjecture!!! *(Can, Sakran)*

Let G be a unipotent linear algebraic group. If S is a complement finite submonoid of the arithmetic submonoid $M = G_{\mathbb{N}}$, then we have
$$\mathbf{d}_M \mathbf{c}_M(S) \leq \mathbf{e}(S)\mathbf{n}_M(S).$$

Thick families in $\mathbf{P}(n, \mathbb{N}_0)$

Let $S \subseteq M = \mathbf{P}(n, \mathbb{N}_0)$ be complement finite submonoid. Let
$$S_j = S \cap (\{0\} \times \cdots \times \mathbb{N}_0 \times \cdots \times \{0\}).$$
Define n_j, c_j and g_j of S_j accordingly.

If $\sum_{j=1}^{n-1} g_j = \mathbf{g}_M(S)$ then S is called **thick** submonoid of M. For example

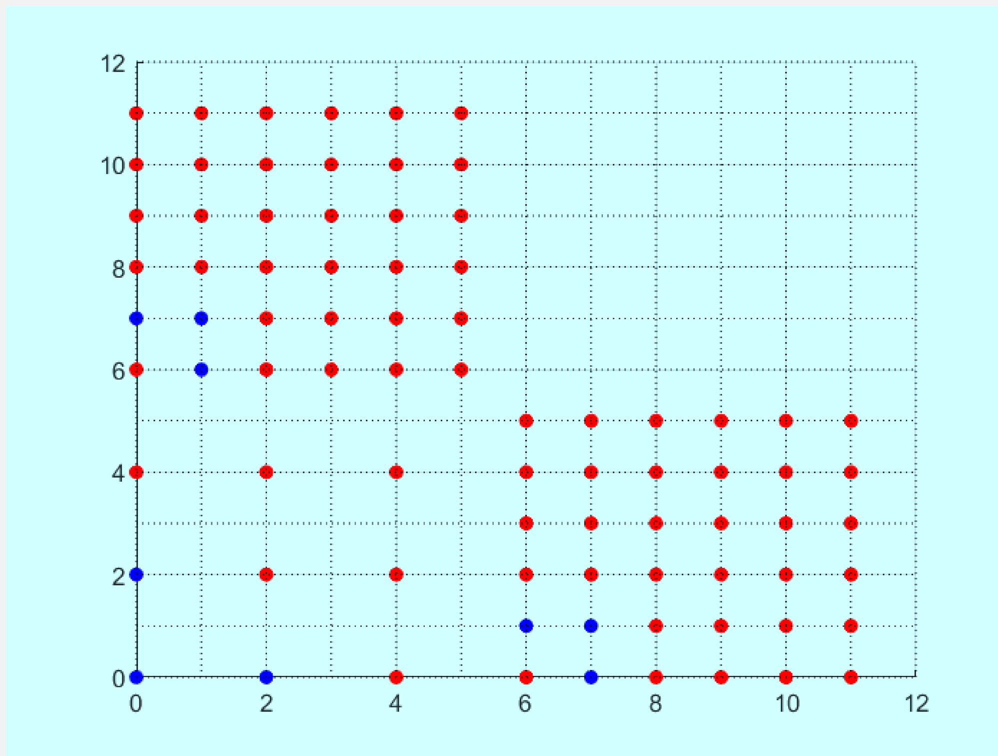


UWC holds for thick family. ***(Can, Sakran)***

Thin families in $\mathbf{P}(n, \mathbb{N}_0)$

If $\prod_{j=1}^{n-1} n_j = \mathbf{n}_M(S)$ then S is called **thin** submonoid of M.

Define $S = \langle \mathbf{1}_2, (2, 0), (0.2), \mathbf{P}_5 \rangle \subseteq M$.



We have $\mathbf{e}(S) = 8$, $\mathbf{n}_M(S) = 9$ and $\mathbf{c}_M(S) = 36$.

If S is thin and $\prod_{j=1}^{n-1} c_j = k^{n-1}$ then **UWC** holds. ***(Can, Sakran)***

Connection with Algebraic Geometry

Let $X = \mathscr{V}(y^2 - x^3 + x)$ be a smooth curve of genus 1. Let $P = (1, 0)$, $Q = (-1, 0) \in X$ and let $\mathfrak{m}_P$ and $\mathfrak{m}_Q$ denote the maximal ideal of $k[X]_P$ and $k[X]_Q$ respectively. Consider the set $H = \{(n_1, n_2) : \exists f \in k(X), (f)_\infty = n_1 P + n_2 Q\}$. Here $(f)_\infty = \mathrm{ord}_P(h)P + \mathrm{ord}_Q(h)Q$ where $f = \frac{g}{h} \in k(X)$ and
$$\mathrm{ord}_P(h) := \max\{k \ : \ h \in \mathfrak{m}_P^k, \ h \notin \mathfrak{m}_P^{k+1}\}$$

From [3], we have that H is a complement finite submonoid of $\mathbb{N}_0^2$ with $|\mathbb{N}_0^2 \backslash H| = 2$. In general, for smooth curve X of genus g, we have
$$\binom{g+2}{2} - 1 \leq |\mathbb{N}_0^2 \backslash H| \leq \binom{g+2}{2} + \frac{(g+1)(g-2)}{2}.$$

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