k-Positivity of the Dual Canonical Basis

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Combinatorial Algebra meets Algebraic Combinatorics

January 22, 2023

Outline

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- k-Positivity

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- Immanants and the Dual Canonical Basis

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So A is totally positive.

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- Has nice topology (Hersh, 2013)

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These matrices have many similar but slightly weaker properties to totally positive matrices.

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Question

What if only a subset of the generalized minors are positive? What does this tell us about the positivity of the rest of the dual canonical basis elements?

Our setting: $G = SL_n$, generalized minors are just minors, we require the set of minors of size less than or equal to k be positive.

Immanants

Let $X = (x_{ij})$ be the matrix of indeterminates.

Definition

Given a function $f : S_n \to \mathbb{C}$, the *immanant* associated to f, $\mathrm{Imm}_f : \mathrm{Mat}_{n\times n}(\mathbb{C}) \to \mathbb{C}$, is the function

$$\mathrm{Imm}_f(X) := \sum_{w\in S_n} f(w)\, x_{1,w(1)} \cdots x_{n,w(n)}.$$

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The most commonly studied immanants are irreducible character immanants ($f = \chi^\lambda$ an irreducible character of S_n).

Kazhdan-Lustzig Immanants

Definition

Let $v \in S_n$. The *Kazhdan–Lusztig immanant* indexed by v is

$$\mathrm{Imm}_v(X) := \sum_{w \in S_n} (-1)^{\ell(w)-\ell(v)} P_{w_0 w, w_0 v}(1)\, x_{1,w(1)} \cdots x_{n,w(n)}$$

where $P_{x,y}(q)$ is the Kazhdan-Lusztig polynomial associated to $x, y \in S_n$ and $w_0 \in S_n$ is the longest permutation.

Row and Column Indices

Let $M = (m_{ij})$ be an $m \times m$ matrix,
$R = \{r_1 \leq \cdots \leq r_n\}, C = \{c_1 \leq \cdots \leq c_n\} \in \left(\!\binom{[m]}{n}\!\right)$.

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We define $M(R, C)$ to be the matrix with element m_{r_i, c_j} in row i, column j.

Example: Let $R = \{1, 1, 3\}, C = \{2, 3, 4\}$.

$$M = \begin{bmatrix} 22 & 18 & 6 & 3 \\ 8 & 7 & 3 & 2 \\ 2 & 2 & 1 & 2 \\ 1 & 2 & 2 & 6 \end{bmatrix}, \quad M(R, C) = \begin{bmatrix} 18 & 6 & 3 \\ 18 & 6 & 3 \\ 2 & 1 & 2 \end{bmatrix}$$

Kazhdan-Lustzig Immanants and the Dual Canonical Basis

Theorem (Skandera, 2008)

The dual canonical basis of $\mathbb{C}[SL_m]$ consists of the nonzero elements of the set $\{\operatorname{Imm}_v X(R,C) \mid v \in S_n \text{ for some } n \in \mathbb{N} \text{ and } R, C \in \left(\!\!\binom{[m]}{n}\!\!\right)\}$.

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Pattern Avoidance

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Is w 321-avoiding?

No: w contains the subsequence 531 and $5 > 3 > 1$.

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Is w 321-avoiding?

No: w contains the subsequence 531 and $5 > 3 > 1$.

Is w is 3412-avoiding?

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Ex: $w = 25341$

Is w 321-avoiding?

No: w contains the subsequence 531 and $5 > 3 > 1$.

Is w is 3412-avoiding?

Yes: length 4 subsequences are 2534, 2531, 2541, 2341, 5341.

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Theorem (C.–Sherman-Bennet, 2023+)

Let $v \in S_n$ be 1324-, 2143-avoiding and suppose that for all $i < j$ with $v(i) < v(j)$ we have $j - i \leq k$ or $v(j) - v(i) \leq k$. Let $R, C \in \left(\!\binom{[m]}{n}\!\right)$. Then $\mathrm{Imm}_v X(R, C)$ is identically 0 or it is k-positive.

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Corollary (C.–Sherman-Bennet, 2023+)

The elements of the dual canonical basis of $\mathbb{C}[SL_m]$ described above are k-positive.

Restriction of a Matrix to Γ_P

Definition

For $P \subseteq S_n$, define Γ_P to be the $n \times n$ array with dots in entries $\{(i, w(i)) \mid w \in P\}$. Let $M|_{\Gamma_P}$ be the matrix M with entries changed to 0 wherever there is no dot in Γ_P.

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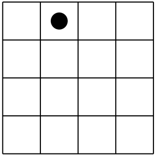
$\Gamma_{[v,w_0]} =$ 

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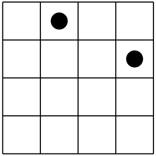
$\Gamma_{[v,w_0]} =$ 

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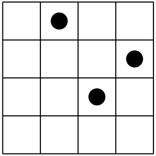
$\Gamma_{[v,w_0]} =$ 

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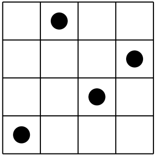

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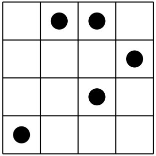

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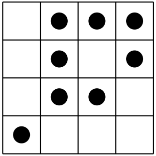

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$$M|_{\Gamma_{[v,w_0]}} = \begin{bmatrix} 0 & 18 & 6 & 3 \\ 0 & 7 & 3 & 2 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Proof Sketch

Proposition (C.–Sherman-Bennet, 2020)

If $v \in S_n$ is 1324-, 2143-avoiding then

$$\mathrm{Imm}_v\, X(R, C) = (-1)^{\ell(v)} \det(X(R, C)|_{\Gamma_{[v,w_0]}}).$$

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We then used Lewis Carroll's identity to do induction.

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- Extend result from 1324-, 2143-avoiding permutations to a larger class.

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- Explore the relationship between these immanants and cluster algebras.

Thank you!

S. Chepuri and M. Sherman-Bennett, *1324- and 2143-avoiding Kazhdan-Lusztig immanants and k-positivity*, Canadian Journal of Mathematics (2021), 1-33.

S. Chepuri and M. Sherman-Bennett, *k-positivity of dual canonical basis elements from 1324-and 2143-avoiding Kazhdan-Lusztig immanants*, preprint (2021), arXiv:2106.09150.