Math 2341, Homework No. 1

Due: Monday October 4th, 2004 in class

1. Let X be a set and V a vector space over a field $\mathbf{K}$. We consider the set $\mathbf{F}(X,V)$ of all functions f from X to V. Given two functions f and g from X to V and a scalar $c \in \mathbf{K}$, we define $f+g\colon X \to V$ and $cf\colon X \to V$, respectively, as $(f+g)(x) = f(x)+g(x)$ and $(cf)(x) = cf(x)$ for all $x \in X$. To show that $\mathbf{F}(X,V)$ is a vector space over K, one should verify 8 axioms. Please show that the following hold:

(1) $f+(g+h) = (f+g)+h$ for $f,g,h \in \mathbf{F}(X,V)$,

(2) there exists an element $\mathcal{O}$ of $\mathbf{F}(X,V)$ such that $f+\mathcal{O} = f$ for all $f \in \mathbf{F}(X,V)$,

(3) $a(bf) = (ab)f$ for all $a,b \in \mathbf{K}$ and $f \in \mathbf{F}(X,V)$.

2. In each case, determine whether or not the set U is a subspace of the given vector space V. Justify your response.

(a) $U = \left\{ A \in \mathbf{M}_{22}(\mathbf{R}) \ ; \ A \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} A \right\}$,
$V = \mathbf{M}_{22}(\mathbf{R})$;

(b) $U = \{ f\colon (0,1) \longrightarrow \mathbf{R} \ ; \ f''(t) \text{ exists and } f''(t) + 2tf'(t) + tf(t) = 0 \ \forall t \in (0,1) \}$,
$V = \mathbf{F}((0,1),\mathbf{R})$.

3. Determine if the matrix $M = \begin{pmatrix} -1+i & -6-i \\ -5+5i & 3+3i \end{pmatrix}$ is a linear combination of the matrices

$$A = \begin{pmatrix} 1+i & 2i \\ 1 & -1+i \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1+i \\ 2i & 2-i \end{pmatrix} \text{ and } C = \begin{pmatrix} 1-i & -i \\ -2 & 1+i \end{pmatrix}.$$

If yes, describe M as a linear combination of these three matrices.

4. Determine a generating set for the solution set of the following system of homogeneous linear equations in $\mathbf{R}^4$:

$$\begin{cases} 2x - 4y - z - w = 0 \\ x - 2y + z + 4w = 0 \\ -2x + 4y - z - 5w = 0 \end{cases}$$

5. Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ 2 & -3 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix}$, and let $U = \mathrm{Vect}_{\mathbf{R}}(A,B,C)$ be the subspace $\mathbf{M}_{2,2}(\mathbf{R})$ generated by A, B and C. Determine the conditions on the real numbers a,b,c,d under which the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ belongs to U.

6. Let V be the vector space of functions from $(0,1)$ to $\mathbf{R}$. Let f, g and h be functions in V given, respectively, by $f(t) = 1/t$, $g(t) = 1/(t-1)$ and $h(t) = 1/(t^2-t)$ for all $t \in (0,1)$. Determine whether these functions are linearly independent. If this is not the case, give a linear dependence relation between them.