Math 2341, Homework No. 3

Due: Monday November 8, 2004 in class

1. Explain why there exists a unique solution to the linear transformation $T\colon \mathbf{R}^2 \to \mathbf{M}_{2,2}(\mathbf{R})$ that satisfies

$$T(1,2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T(2,3) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Justify your answer clearly, indicating what theorem you have used.
For this linear application

(i) Give a formula for the image $T(x,y)$ of any point (x,y) of $\mathbf{R}^2$;

(ii) Use part (i) to compute $T(1,1)$.

2. For each of the linear transformations T below, determine the dimension of $\ker(T)$ and that of $\operatorname{Im}(T)$. Use this information to conclude whether T is injective, surjective, or bijective. If T is bijective give a formula for T^{-1}.

(i) $T\colon \mathbf{P}_2(\mathbf{R}) \to \mathbf{R}^3$ given by $T(p(x)) = (p(-1), p(0), p(1))$.

(ii) $T\colon \mathbf{M}_{2,2}(\mathbf{R}) \to \mathbf{M}_{2,2}(\mathbf{R})$ given by $T(A) = A + A^t$.

3. Let $\mathbf{P}(\mathbf{R})$ be the vector space of all polynomials $p(x)$ with coefficients in $\mathbf{R}$ (with no restrictions on the degree of $p(x)$). We consider the linear applications D, S and T of $\mathbf{P}(\mathbf{R})$ to $\mathbf{P}(\mathbf{R})$ given by

$$D(p(x)) = p'(x) = \frac{dp}{dx}(x), \quad T(p(x)) = (x+1)p(x) \quad \text{and} \quad S(p(x)) = p(2x+1).$$

Give formulas for

(i) $(2T - S)(p(x))$ (ii) $(T \circ S)(p(x))$ (iii) $(S \circ T)(p(x))$ (iv) $(T - (D \circ T))(p(x))$.

4. Let $R\colon \mathbf{R}^3 \to \mathbf{R}^3$ be a rotation of $\pi/3$ around the x-axis in $\mathbf{R}^3$. Find the matrix $[R]_{\mathcal{E}}$ that defines the linear transformation R in the standard basis $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbf{R}^3$. Find $R(1,2,1)$.

5. Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation given by $T(x,y) = (x-y, x+y)$ for all $(x,y) \in \mathbf{R}^2$. We consider the bases $\mathcal{E} = \{(1,0),(0,1)\}$ et $\mathcal{B} = \{(1,1),(1,-1)\}$ of $\mathbf{R}^2$. Determine the matrices $[T]_{\mathcal{E}}$ and $[T]_{\mathcal{B}}$ that represent T in each one of these bases.