MAT 1341B: Introduction to Linear Algebra

Instructor: Erhard Neher

Mini test 2 (November 19, 2004)

Family Name: ______________________________

First Name: ______________________________

Student number: ___________________________

Please read these instructions carefully:

- This is a closed book exam, and no notes of any kind are allowed. **The use of calculators, cell phones, pagers or any text storage or communication device is not permitted, and will be considered as fraud.**
- The exam has 5 multiple choice questions and 3 questions requiring detailed answers. The multiple choice questions are each worth 3 points each and no part marks will be given. **Please record your answers on the line "My answer: ___________" provided for each multiple choice question.**
- The last three questions are long-answer questions, and partial marks may be earned. Please be careful to include all details, and explain what you are doing. Question 7 depends on your student number. You must use the correct value to receive credit for the question.
- Read each question carefully - you will save yourself time and unnecessary grief later on. Where it is possible to check your work, do so.
- If you do not have enough space, use the back of the pages and clearly indicate this. The exam has 9 pages. You have 80 minutes to complete this exam.

Good luck!

Quest.	total MC	6	7.	8.	Total
maximal	15	6	6	3	30
answer					

1. Which of the following are subspaces?

$$U = \{ \begin{bmatrix} -s & t & 2 & 0 \end{bmatrix}^T \in \mathbb{R}^4 \mid s, t \in \mathbb{R} \}$$
$$V = \{ \begin{bmatrix} s & t & 2s & t \end{bmatrix}^T \in \mathbb{R}^4 \mid s, t \in \mathbb{R} \}$$
$$W = \{ \begin{bmatrix} r & s & t \end{bmatrix}^T \in \mathbb{R}^3 \mid 2r - 5s + 7t = 0 \}$$

A. V only.
B. U only.
C. W only.
D. V and W only.
E. U and W only.
F. U and V only.
G. All three of them.

My answer:____________

Answer: U does not contain the zero vector $0 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T \in \mathbb{R}^4$, and is therefore not a subspace. The set V is a span, namely $V = \mathit{span}\,\{\begin{bmatrix} 1 & 0 & 2 & 0 \end{bmatrix}^T, \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}^T\}$ and is therefore a subspace. Also W is a subspace, since $W = \mathit{null}\,A$ for $A = \begin{bmatrix} 2 & -5 & 7 \end{bmatrix}$. For V and W one can also easily check the three conditions defining a subspace. The correct answer is therefore D. **Reference**: suggested exercise §4.1, 5bdf, and 5ace done in the problem session of Oct. 29.

2. (3 points) The eigenvalues of the matrix

$$\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$$

are

A. $1 \pm \sqrt{2}\, i$.
B. $2 \pm \sqrt{3}$.
C. $2 \pm i$.
D. $3 \pm 4i$
E. $2 \pm \sqrt{2}\, i$
F. $1 \pm i$

My answer:____________

Answer: More generally, the characteristic polynomial of any matrix of the form $\begin{pmatrix} a & b \\ c & a \end{pmatrix}$ is $c_A(x) = (x-a)^2 - bc$, where a, b, c can be complex numbers. The eigenvalues are the zeros of the quadratic polynomial $c_A(x)$, hence are the solutions of $(x-a)^2 = bc$ which are $a \pm \sqrt{bc}$. In the problem above we have $a = 1$, $b = 2$, $c = -1$. The eigenvalues are therefore $1 \pm \sqrt{-2} = 1 \pm \sqrt{2}i$, i.e., the correct answer is A. **Reference**: suggested exercise §2.5, 12bd; Problem session of Oct. 29.

3. (3 points) Which of the following assertions are true for vectors $X_i \in \mathbb{R}^n$?

(i) If $3X_1 + 2X_2 - 5X_3 = 0$ then X_1, X_2, X_3 are linearly independent.

(ii) If none of the X_i are zero, then X_1, X_2, X_3 are a basis of *span* $\{X_1, X_2, X_3\}$.

(iii) If X_1, X_2, X_3 and X_4 are a basis of a subspace of $\mathbb{R}^n$ then $n \geq 4$.

(iv) Every nonzero subspace of $\mathbb{R}^n$ has a basis.

(v) If a subspace contains a spanning set of m vectors and k linearly independent vectors, then $k > m$.

A. (i) and (ii) only.
B. (i) and (iv) only.
C. (ii), (iii) and (v).
D. (ii) and (v) only.
E. (iii), (iv) and (v).
F. (iii) and (iv) only.

My answer:____________

Answer: (i) and (ii) are false, (iii) is correct (follows for example from §4.3, Theorem 4), (iv) is correct (§4.3, Theorem 3), (v) is false (contradicts §4.3, Theorem 1). Thus the correct answer is F. **Reference:** for (i) and (ii): suggested exercise §4.2, 7df and 7e, done in problem session on Nov. 5.

4. (3 points) Find the projection $proj_U(X)$ for

$$U = span\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\} \quad \text{and} \quad X = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}.$$

A. $\frac{1}{2}\begin{bmatrix} 2 & -1 & 2 & -1 \end{bmatrix}^T$.
B. $\frac{1}{2}\begin{bmatrix} 2 & 1 & 2 & 1 \end{bmatrix}^T$.
C. $\frac{1}{2}\begin{bmatrix} 1 & 3 & 1 & 3 \end{bmatrix}^T$.
D. $\frac{1}{2}\begin{bmatrix} 2 & 3 & 2 & 3 \end{bmatrix}^T$.
E. $\frac{1}{2}\begin{bmatrix} -1 & 3 & -1 & 3 \end{bmatrix}^T$.
F. $\frac{1}{2}\begin{bmatrix} 2 & 0 & 2 & 0 \end{bmatrix}^T$.

My answer:___________

Answer: Let $F_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$ and $F_2 = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T$. Since $F_1 \cdot F_2 = 0$ we have an orthogonal basis of U. For arbitrary $X = \begin{bmatrix} a & b & c & d \end{bmatrix}^T$ we then get

$$\begin{aligned} proj_U(X) &= \frac{X \cdot F_1}{F_1 \cdot F_1} F_1 + \frac{X \cdot F_2}{F_2 \cdot F_2} F_2 \\ &= \frac{1}{4}(a+b+c+d)\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T + \frac{1}{4}(a-b+c-d)\begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T \\ &= \frac{1}{2}\begin{bmatrix} a+c & b+d & a+c & b+d \end{bmatrix}^T. \end{aligned}$$

For the given X we have $a = 1 = c = d$ and $b = 2$, hence $a+c = 2$, $b+d = 3$ so the solution is D. **Reference**: suggested exercise §4.6, 1b and 2b, and problem done in class on Nov. 16.

5. (3 points) The general solution of the system of linear differential equations

$$\begin{aligned} f_1' &= -f_1 + f_2 \\ f_2' &= 4f_1 + 2f_2 \end{aligned}$$

is of the form below, where c, $d \in \mathbb{R}$ are arbitrary:

A. $ce^{3x}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + de^{2x}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

B. $ce^{3x}\begin{bmatrix} 1 \\ 4 \end{bmatrix} + de^{-2x}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

C. $ce^{3x}\begin{bmatrix} -4 \\ 1 \end{bmatrix} + de^{-2x}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

D. $ce^{3x}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + de^{-2x}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

E. $ce^{-3x}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + de^{-2x}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

F. $ce^{-3x}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + de^{2x}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

My answer:____________

Answer: The system has the form $f' = Af$ for $A = \begin{bmatrix} -1 & 1 \\ 4 & 2 \end{bmatrix}$. The eigenvalues λ_i of A and their corresponding eigenvectors X_i are $\lambda_1 = 3$, $X_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\lambda_2 = -2$, $X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. By §2.8, Theorem 1, the general solution is therefore given in B. **Reference**: suggested exercise §2.8, 1bd, and 1c done in the problem session on Oct. 29.

6. (6 points) Find a basis of the subspace

$$U = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \mid 3a - b + c + d = 0 \right\}$$

of $\mathbb{R}^4$. Justify your answer, i.e., you must either show that you have a basis or quote some theorems from class.

Answer: Note that

$$U = \left\{ \begin{bmatrix} a \\ 3a+c+d \\ c \\ d \end{bmatrix} \mid a, c, d \in \mathbb{R} \right\} = \left\{ a \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \mid a, c, d \in \mathbb{R} \right\}$$
$$= span \left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

The set $\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is therefore a spanning set of U. It is also linearly independent. This can be seen from

$$a \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = 0 \quad \Rightarrow \quad \begin{bmatrix} a \\ 3a+c+d \\ c \\ d \end{bmatrix} = 0 \quad \Rightarrow \quad a = c = d = 0$$

or by noticing that the 3×4-matrix whose rows are the three vectors is in row-echelon form and therefore has linearly independent rows. **Reference**: suggested exercise §4.3, 7bd and 7e, done in the problem session of Nov. 5.

7. (6 points) In the matrix A below replace α with the **last** digit of your student number, write the new matrix obtained in this way next to the given A:

$$A = \begin{bmatrix} 1 & -2 & 0 & 3 \\ 3 & 2 & 8 & 9 \\ 2 & 3 & 7 & \alpha \\ -1 & 3 & 1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 & 3 \\ 3 & 2 & 8 & 9 \\ 2 & 3 & 7 & \dots \\ -1 & 3 & 1 & -3 \end{bmatrix}.$$

Find

(i) a basis for the row space of A,
(ii) a basis for the column space of A, and
(iii) the rank of A.

Answer: We use elementary row operations to find a row-echelon form of A:

$$\begin{bmatrix} 1 & -2 & 0 & 3 \\ 3 & 2 & 8 & 9 \\ 2 & 3 & 7 & \alpha \\ -1 & 3 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 8 & 8 & 0 \\ 0 & 7 & 7 & \alpha-6 \\ 0 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 7 & 7 & \alpha-6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & \alpha-6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\alpha = 6$: Then a row-echelon form of A is

$$R = \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and the answers to the questions are: (i) A basis of the row space of A is given by the non-zero rows of R, i.e., $\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$. (ii) A basis of the column space of A is given by the columns of A corresponding to the columns of R with leading 1's, i.e., a basis is $\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 3 \\ 3 \end{bmatrix} \right\}$. (iii) The rank of A is 2.

$\alpha \neq 6$: Then a row-echelon form of A is

$$R = \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and the answers to the questions are: (i) A basis of the row space of A is given by the non-zero rows of R, i.e., $\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. (ii) A basis of the column space of A is given by the columns of A corresponding to the columns of R with leading 1's, i.e., a basis is $\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ \alpha \\ -3 \end{bmatrix} \right\}$.
(iii) The rank of A is 3.
Reference: suggested exercise §4.4, 5b, and 5a, done in the problem session of Nov. 12.

Marking: finding a row-echelon form **(3 points)**, interpreting the result correctly; (i) **(1 points)**, (ii) **(1 points)** , iii) **(1 points)**.

8. (3 points) Prove that a set $\{X_1, \ldots, X_m\}$ of orthogonal vectors in $\mathbb{R}^n$ is linearly independent.

Answer: Suppose $t_1X_1 + \cdots + t_mX_m = 0$, and let i be a number between 1 and m. Since $X_i \cdot X_j = 0$ for $i \neq j$ we get

$$\begin{aligned} 0 &= X_i \cdot 0 = X_i \cdot (t_1X_1 + \cdots + t_mX_m) \\ &= t_1X_i \cdot X_1 + \cdots + t_{i-1}X_i \cdot X_{i-1} + t_iX_i \cdot X_i + t_{i+1}X_i \cdot X_{i+1} + \cdots + t_mX_i \cdot X_m \\ &= t_iX_i \cdot X_i \end{aligned}$$

But $X_i \cdot X_i > 0$ because $X_i \neq 0$ by assumption. Therefore $t_i = 0$. Since i was arbitrary, this shows that $t_1 = t_2 = \cdots = t_m = 0$. Hence $\{X_1, \ldots, X_m\}$ is linearly independent. **Reference**: Done in class on Nov. 11 (lecture 18).