SOLUTIONS TO MATH 1341 , TEST 3 (YELLOW EXAM)

1. The set U is not closed under addition: For example $X = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^T$ and $Y = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$ lie in U, but their sum $X + Y = \begin{bmatrix} 0 & 2 & 0 \end{bmatrix}^T$ does not. The set V is a subspace, since it is closed under scalar multiplication and addition, and it contains the zero vector. W does not contain the zero vector $0 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T \in \mathbb{R}^4$, and is therefore not a subspace. The correct answer is therefore B.

   Reference: suggested exercise §4.1, 5bdf, and 5ace.

2. The determinant of A is i, hence $\det A \neq 0$ and so A is invertible (hence D is wrong). Its inverse is given by
$$A^{-1} = \frac{1}{i}\begin{bmatrix} i & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1/i \\ 0 & 1/i \end{bmatrix} = \begin{bmatrix} 1 & -i \\ 0 & -i \end{bmatrix}.$$
   So answer B is false too. The matrix A is a triangular matrix, hence its eigenvalues are just the entries of the diagonal, i.e., i and 1. Thus, A has two distinct eigenvalues and is therefore diagonalizable (so A and C are wrong). A matrix with a nonzero determinant need not be diagonalizable, so E is false. The correct answer is F.

   Reference: suggested exercise §2.5, 11, 12.

3. (ii) is wrong: $A = I_n$ fulfills the condition but has only one eigenvalue, namely 1. The given condition is one of the conditions of the Invertible Matrix Theorem (precisely: condition (3)), hence A is invertible, and therefore (i), (iii) and (iv) are true: They are just conditions (9), (6) and (15) of the Invertible Matrix Theorem. The correct answer is therefore E.

4. (i) is not correct, as we have $\mathrm{rank A} \leq 2$ since there can be at most one leading 1 in every row. By Theorem 6 of §4.4 we then get $\dim\ker \mathrm{A} = 6 - \mathrm{rank A} \geq 4$. (ii) is not correct, since any subspace U of $\mathbb{R}^2$ has a spanning set of 3 vectors: simply take a basis of the subspace. This will have $\dim U$ vectors, with $\dim U \leq 2$. We can add $3 - \dim U$ vectors in U to get a spanning set of U with 3 vectors, which of course will not be a basis. (iii) is correct, see Th. 3 in §4.3. (iv) is correct: by the fundamental Theorem and its Corollary (§4.3), as any 4 nonzero vectors in $\mathbb{R}^3$ are linearly dependent. The correct answer is therefore F.

5. Let $F_1 = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix}^T$ and $F_2 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$. Since $F_1 \cdot F_2 = 0$ we have an orthogonal basis of U. For arbitrary $X = \begin{bmatrix} a & b & c \end{bmatrix}^T$ we then get
$$\begin{aligned} proj_U(X) &= \tfrac{X \cdot F_1}{F_1 \cdot F_1} F_1 + \tfrac{X \cdot F_2}{F_2 \cdot F_2} F_2 \\ &= \tfrac{1}{6}(a + 2b - c)\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \tfrac{1}{2}(a + c)\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\ &= \tfrac{1}{6}\begin{bmatrix} 4a + 2b + 2c \\ 2a + 4b - 2c \\ 2a - 2b + 4c \end{bmatrix}. \end{aligned}$$
   For the given X we have $a = 1, b = 2, c = 3$,
$$proj_U(X) = \frac{1}{6}\begin{bmatrix} 4 + 4 + 6 \\ 2 + 8 - 6 \\ 2 - 4 + 12 \end{bmatrix} = \begin{bmatrix} 7/3 \\ 2/3 \\ 5/3 \end{bmatrix}.$$

So the solution is E.

Reference: suggested exercise §4.6, 1b and 2b.

6. By definition, $F_1 = X_1$. The vector F_2 is determined by

$$F_2 = X_2 - \frac{X_2 \cdot F_1}{F_1 \cdot F_1} F_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

and F_3 is given by

$$F_3 = X_3 - \frac{X_3 \cdot F_1}{F_1 \cdot F_1} F_1 - \frac{X_3 \cdot F_2}{F_2 \cdot F_2} F_2 = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} - \frac{4}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}.$$

Marking: 1 point for each part of the formula (i.e. for each F_i) for a total of 3 points, 1 point for the correct answer of each of F_2 and F_3.

7. Note that

$$\begin{aligned} U &= \left\{ \begin{bmatrix} a \\ a - 3c \\ c \\ -2a + 5c \end{bmatrix} \mid a, c \in \mathbb{R} \right\} \\ &= \left\{ a \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix} + c \begin{bmatrix} 0 \\ -3 \\ 1 \\ 5 \end{bmatrix} \mid a, c \in \mathbb{R} \right\} \\ &= \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 1 \\ 5 \end{bmatrix} \right\}. \end{aligned}$$

The set $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 1 \\ 5 \end{bmatrix} \right\}$ is therefore a spanning set of U. It is also linearly independent. This can be seen from

$$a \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix} + b \begin{bmatrix} 0 \\ -3 \\ 1 \\ 5 \end{bmatrix} = 0 \quad \Longrightarrow \quad \begin{bmatrix} a \\ a - 3b \\ c \\ -2a + 5b \end{bmatrix} = 0 \quad \Longrightarrow \quad a = b = 0$$

or by noticing that the 2×4-matrix whose rows are the two vectors is in row-echelon form and therefore has linearly independent rows.

Marking: 2 points for finding the spanning set, 2 points for linear independence.

Reference: suggested exercise §4.3, 7bd and 7e.

8. We use elementary row operations to find a row-echelon form of A:

$$\begin{bmatrix} 1 & 1 & 2 & -1 \\ -1 & -1 & -1 & 3 \\ 3 & 3 & 9 & \alpha \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & \alpha+3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & \alpha-3 \end{bmatrix}.$$

$\alpha = 3$: Then a row-echelon form of A is

$$R = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and the answers to the questions are:

(i) A basis of the row space of A is given by the non-zero rows of R, i.e.,

$$\{\begin{bmatrix} 1 & 1 & 2 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 2 \end{bmatrix}\}.$$

(ii) A basis of the column space of A is given by the columns of A corresponding to the columns of R with leading 1's, i.e., a basis is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 9 \end{bmatrix} \right\}.$$

(iii) The rank of A is 2.

$\alpha \neq 3$: Then a row-echelon form of A is

$$R = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and the answers to the questions are:

(i) A basis of the row space of A is given by the non-zero rows of R, i.e.,

$$\{\begin{bmatrix} 1 & 1 & 2 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}\}.$$

(ii) A basis of the column space of A is given by the columns of A corresponding to the columns of R with leading 1's, i.e., a basis is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 9 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ \alpha \end{bmatrix} \right\}.$$

(iii) The rank of A is 3.

Reference: suggested exercise §4.4, 5b, and 5a.

Marking: finding a row-echelon form (3 points), interpreting the result correctly; (i) (1 point), (ii) (1 point), (iii) (1 point).

9. Let $X \in U^{\perp}$. This means that $X \cdot Y = 0$ for all $Y \in U$, hence in particular for all X_i, $1 \leq i \leq m$ since all $X_i \in U$. This proves that $U^{\perp} \subset \{X \in \mathbb{R}^m \mid X \cdot X_i = 0 \text{ for } 1 \leq i \leq m\}$.

Conversely, suppose $X \in \mathbb{R}^n$ has the property that $X \cdot X_i = 0$ for all $1 \leq i \leq m$, and let $Y \in U$. Since $U = \mathrm{span}\{\mathrm{X_1, \ldots, X_m}\}$ we can write Y as a linear combination of the X_i, $1 \leq i \leq m$, say $Y = c_1 X_1 + c_2 X_2 + \cdots + c_m X_m$. Then

$$\begin{aligned} X \cdot Y &= X \cdot (c_1 X_1 + c_2 X_2 + \cdots + c_m X_m) \\ &= (X \cdot c_1 X_1) + (X \cdot c_2 X_2) + \cdots + (X \cdot c_m X_m) \\ &= c_1 (X \cdot X_1) + c_2 (X \cdot X_2) + \cdots + c_m (X \cdot X_m) \\ &= c_1\, 0 + c_2\, 0 + \cdots + c_m\, 0 = 0, \end{aligned}$$

which proves that $X \in U^{\perp}$. Hence we also have the other inclusion, $U^{\perp} \supset \{X \in \mathbb{R}^m \mid X \cdot X_i = 0 \text{ for } 1 \leq i \leq m\}$, and therefore the claimed equality.

Marking: 1 point for some relevant justification, 2 points for a proof that is missing only some minor steps, 3 points for correct proof. No points for proof by example.