1. Which of the following statements are true for the matrix $A = \begin{pmatrix} -1 & 3 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ and the vectors $X_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

A. X_2 is an eigenvector of A with eigenvalue 3 but X_1 is not an eigenvector of A.

B. Neither X_1 nor X_2 are eigenvectors of A.

C. X_1 is an eigenvector of A with eigenvalue 2 but X_2 is not an eigenvecor of A.

D. X_1 is an eigenvector of A with eigenvalue -1 and X_2 is an eigenvector of A with eigenvalue 2.

E. X_1 is an eigenvector of A with eigenvalue 2 and X_2 is an eigenvector of A with eigenvalue 3.

F. X_1 and X_2 are both eigenvectors of A with eigenvalue 3.

. . .

2 Find the solutions of the following linear system

$$\begin{cases} x + (1-i)\,y = 2-i \\ ix + \qquad 2\,y = 2+3i \end{cases}$$

What is the value of x?

A. $1+i$ **B.** $-1-i$ **C.** $2-i$
D. $1-2i$ **E.** $-1+2i$ **F.** -2

3. For which values of x do the vectors $(x, 1, 3)$, $(0, 1, 2)$ and $(x, 1, x)$ form a basis of $\mathbb{R}^3$?

A. $x \neq -2$, $x \neq 1$ and $x \neq 3$
B. $x \neq 0$ and $x \neq 3$
C. $x \neq -1$, $x \neq 2$ and $x \neq 3$
D. $x \neq 0$ and $x \neq 1$
E. $x \neq 1$ and $x \neq 3$
F. $x \neq -1$, $x \neq 0$ and $x \neq 3$

4. Find the intersection between the x-axis and the plane passing through the points $(1, 0, 1)$, $(1, 1, 0)$ and $(0, 1, 1)$.

A. $(-1, 0, 0)$ **B.** $(2, 0, 0)$ **C.** $(1, 0, 0)$
D. $(0, 0, 0)$ **E.** $(3, 0, 0)$ **F.** $(-2, 0, 0)$

5 [redacted]. Let A be a 3×7-matrix of rank 3. Which of the following statements are true?

(i) The set of solutions of the homogeneous linear systems $AX = 0$ is a subspace of $\mathbb{R}^7$ of dimension 3.

(ii) The row space of A has dimension 3.

(iii) For every $B \in \mathbb{R}^3$, the linear system $AX = B$ has infinitely many solutions.

A. none of the statements is true
B. (iii) only
C. (i) and (ii)
D. all statements are true
E. (ii) and (iii)
F. (i) only

6. Suppose $\{v, w\}$ is a basis of the vector space V. Which of the following sets are then also bases of V?

(i) $\{v + w, v\}$
(ii) $\{v - w, w - v\}$
(iii) $\{v + w, -v, w\}$

A. (ii) and (iii)
B. all of them
C. (i) only
D. (i) and (ii)
E. (iii) only
F. none of them

7. Paul and Richard participate in a bike race from A to B and then to C. Paul's average speed is 30 km/h between A and B and 40 km/h between B and C, while Richard's average speed is 40 km/h between A and B and 50 km/h between B and C. The total time for going from A to C is 9 hours for Paul and 7 hours for Richard. Find the distance between A and B.

A. 120 km	**B.** 240 km	**C.** 200 km
D. 80 km	**E.** 40 km	**F.** 160 km

8. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation satisfying $T(1,0) = (1,2,1)$ and $T(0,1) = (1,0,1)$. Find $T(1,-2)$.

A. $(0,2,0)$	**B.** $(3,2,3)$	**C.** $(-1,2,-1)$
D. $(2,-1,3)$	**E.** $(0,0,0)$	**F.** $(1,0,3)$

9. Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$, let I be the 2×2 identity matrix and let X be the 2×2-matrix satisfying

$$(AX + I)^T = 2I.$$

Then the first row of X is

A. $(3, -1)$ **B.** $(3, -2)$ **C.** $(2, -1)$
D. $(-1, 1)$ **E.** $(1, 0)$ **F.** $(-2, 1)$

10. Let A, B and C be 2×2 matrices and assume $\det(A) = 2$, $\det(B) = 3$ and $\det(C) = 4$. Then the 2×2 matrix M satisfying $AMB = C$ has the properties

A. $M = B^{-1}CA^{-1}$ and $\det(M) = 2/3$
B. $M = CB^{-1}A^{-1}$ and $\det(M) = 24$
C. $M = B^{-1}CA^{-1}$ and $\det(M) = 24$
D. $M = A^{-1}CB^{-1}$ and $\det(M) = 24$
E. $M = A^{-1}CB^{-1}$ and $\det(M) = 2/3$
F. $M = CB^{-1}A^{-1}$ and $\det(M) = 2/3$

11. (10 pts) Let

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

(i) Find the rank of A and a basis of its row space.

(ii) Find a basis for the null space of A.

12. (10 pts) Let $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and let $U = \{A \in \mathrm{M}_{22}\,;\, AB = 0\}$.

(i) Show that U is a subspace of the vector space M_{22} of 2×2-matrices.

(ii) Find a basis of U. Justify your answer.

13. (12 pts) Consider the polynomials

$$p_1(x) = x + 1, \quad p_2(x) = x^2 - x + 1 \quad \text{and} \quad p_3(x) = x^2 + 1\,.$$

(i) Determine if these polynomials are linearly independent.

(ii) Does the polynomial x^2 lie in the span of $p_1(x)$, $p_2(x)$ and $p_3(x)$? If yes, write x^2 as a linear combination of these three polynomials.

(iii) Is $\{p_1(x), p_2(x), p_3(x)\}$ a basis of the vector space $\mathbf{P}_2$ of polynomials of degree ≤ 2? Justify your answer.

14. (8 pts) Give a basis of the space of solutions of the differential equation

$$f'' - 3f' + 2f = 0$$

and find the solution satisfying $f(0) = 1$ and $f'(0) = 0$.

15. (10 pts) In each case, find a basis and the dimension of the subspace U of the vector space V. Justify your answers.

(i) $V = \mathbb{R}^4$, $U = \{(a+b,\, a+c,\, b,\, c)\,;\, a, b, c \in \mathbb{R}\}$

(ii) $V = \mathbf{P}_2$, $U = \{p(x) \in \mathbf{P}_2\,;\, p(0) - p(1) = 0\}$

16. (10 pts) The eigenvalues of the matrix $A = \begin{pmatrix} -8 & 6 \\ -9 & 7 \end{pmatrix}$ are -2 and 1.

(i) Find a basis of each eigenspace of A.

(ii) Is A diagonalizable? Justify your answer. If your answer is yes, find a matrix P such that $P^{-1}AP$ is diagonal and give $P^{-1}AP$.