1. Which of the following statements are true for the matrix $A = \begin{pmatrix} -1 & 3 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ and the vectors $X_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

A. X_2 is an eigenvector of A with eigenvalue 3 but X_1 is not an eigenvector of A.

B. Neither X_1 nor X_2 are eigenvectors of A.

$\boxed{\textbf{C.}}$ X_1 is an eigenvector of A with eigenvalue 2 but X_2 is not an eigenvecor of A.

D. X_1 is an eigenvector of A with eigenvalue -1 and X_2 is an eigenvector of A with eigenvalue 2.

E. X_1 is an eigenvector of A with eigenvalue 2 and X_2 is an eigenvector of A with eigenvalue 3.

F. X_1 and X_2 are both eigenvectors of A with eigenvalue 3.

$$\begin{pmatrix} -1 & 3 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 3 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 3 \end{pmatrix} \neq \lambda\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

2. Find the solutions of the following linear system

$$\begin{cases} x + (1-i)\,y = 2 - i \\ i\,x + \quad 2y = 2 + 3i \end{cases}$$

What is the value of x?

A. $-1 + 2i$

B. $1 + i$

C. $-1 - i$

D. -2

E. $2 - i$

$\boxed{\textbf{F.}}$ $1 - 2i$

$$\left(\begin{array}{cc|c} 1 & 1-i & 2-i \\ i & 2 & 2+3i \end{array}\right) \sim \begin{pmatrix} 1 & 1-i & 2-i \\ 0 & 2-i(1-i) & 2+3i-i(2-i) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1-i & 2-i \\ 0 & 1-i & 1+i \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1-2i \\ 0 & 1-i & 1+i \end{pmatrix}$$

Corresponding system: $x = 1-2i$

$(1-i)y = 1+i$, ie $y = \frac{1+i}{1-i} = \frac{1}{2}(1+i)^2 = i$

3. For which values of x form the vectors $(x,1,3)$, $(0,1,2)$ and $(x,1,x)$ a basis of $\mathbb{R}^3$?

A. $x \neq -1$, $x \neq 0$ and $x \neq 3$

$\boxed{\textbf{B.}}$ $x \neq 0$ and $x \neq 3$

C. $x \neq -1$, $x \neq 2$ and $x \neq 3$

D. $x \neq 0$ and $x \neq 1$

E. $x \neq -2$, $x \neq 1$ and $x \neq 3$

F. $x \neq 1$ and $x \neq 3$

The vectors form a basis if and only if the matrix consisting of the vectors is invertible:

$$\begin{vmatrix} x & 1 & 3 \\ 0 & 1 & 2 \\ x & 1 & x \end{vmatrix} = \begin{vmatrix} x & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & x-3 \end{vmatrix} = x(x-3)$$

Therefore, they form a basis $\Leftrightarrow x(x-3) \neq 0$

$\Leftrightarrow x \neq 0$ and $x \neq 3$

4. Find the intersection between the x-axis and the plane passing through the points $(1,0,1)$, $(1,1,0)$ and $(0,1,1)$.

A. $(3,0,0)$

$\boxed{\textbf{B}}$ $(2,0,0)$

C. $(0,0,0)$

D. $(-2,0,0)$

E. $(-1,0,0)$

F. $(1,0,0)$

The plane passing through the 3 given points has the form $ax+by+cz=d$ where a,b,c,d satisfy

$$\begin{cases} a + c = d \\ a+b = d \\ b+c = d \end{cases} \Leftrightarrow \begin{cases} a + c = d \\ b-c=0 \\ b+c=d \end{cases} \Leftrightarrow \begin{cases} a+c=d \\ b=c \\ 2b=d \end{cases}$$

$$\Leftrightarrow b = \frac{d}{2} = c = a$$

Hence, an equation of the plane is $x+y+z=2$

The x-axis is determined by the equations $y=0=z$, hence the intersection of the x-axis and the plane satisfies

$$\left.\begin{array}{r} x+y+z=2 \\ y=0 \\ z=0 \end{array}\right\} \Rightarrow \begin{array}{l} x=2 \\ y=0 \\ z=0 \end{array}$$

5. Let A be a 3×7-matrix of rank 3. Which of the following statements are true?

(i) The set of solutions of the homogeneous linear systems $AX = 0$ is a subspace of $\mathbb{R}^7$ of dimension 3.

(ii) The row space row$A$ has dimension 3.

(iii) For every $B \in \mathbb{R}^3$, the linear system $AX = B$ has infinitely many solutions.

A. (i) only

B. (i) and (ii)

C. (iii) only

$\boxed{\textbf{D.}}$ (ii) and (iii)

E. all statements are true

F. none of the statements is true

(i) false : $\dim \operatorname{null} A = 7 - 3 = 4$

(ii) $\dim \operatorname{row} A = \operatorname{rank} A = 3$, so (ii) is true

(iii) true since the corresponding augmented matrix has ref

$$\begin{pmatrix} 1 & & & * \\ 0 & 1 & & \\ 0 & 0 & 1 & \cdots \end{pmatrix}$$

6. Suppose $\{\vec{v}, \vec{w}\}$ is a basis of the vector space V. Which of the following sets are then also bases of V?

(i) $\{\vec{v} + \vec{w},\ \vec{v}\}$

(ii) $\{\vec{v} - \vec{w},\ \vec{w} - \vec{v}\}$

(iii) $\{\vec{v} + \vec{w},\ -\vec{v},\ \vec{w}\}$

A. (ii) and (iii)

B. none of them

$\boxed{\textbf{C.}}$ (i) only

D. (iii) only

E. (i) and (ii)

F. all of them

(i) is a basis since $\dim V = 2$ and $\{v+w, v\}$ are linearly independent: $0 = s(v+w) + tv =$
$= (s+t)v + sw \Rightarrow s = 0 = t$ since $\{v, w\}$ lin. indep.

(ii) is not a basis since $\{v-w, w-v\}$ are linearly dependent: $(v-w) + (w-v) = 0$

(iii) is not a basis, since any basis of a 2-dimensional space has exactly 2 elements.

7. Paul and Richard participate in a bike race from A to B and then to C. Paul's average speed is 30 km/h between A and B and 40 km/h between B and C, while Richard's average speed is 40 km/h between A and B and 50 km/h between B and C. The total time for going from A to C is 9 hours for Paul and 7 hours for Richard. Find the distance between A and B.

A. 160 km

B. 80 km

$\boxed{\textbf{C.}}$ 120 km

D. 240 km

E. 40 km

F. 200 km

Let x = distance from A to B and y = distance from B to C. Then, Paul's data give $\frac{x}{30}+\frac{y}{40}=9$, while Richard's data yield $\frac{x}{40}+\frac{y}{50}=7$. Thus, we have to solve the linear system:

$$\begin{pmatrix} \frac{1}{30} & \frac{1}{40} & 9 \\ \frac{1}{40} & \frac{1}{50} & 7 \end{pmatrix} \sim \begin{pmatrix} 4 & 3 & 9\cdot 120 \\ 5 & 4 & 7\cdot 200 \end{pmatrix} = \begin{pmatrix} 4 & 3 & 1080 \\ 5 & 4 & 1400 \end{pmatrix}$$

$$\sim \begin{pmatrix} 4 & 3 & 1080 \\ 1 & 1 & 320 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 320 \\ 4 & 3 & 1080 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 320 \\ 1 & 0 & 1080-3\cdot 320 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 320 \\ 1 & 0 & 120 \end{pmatrix}$$

Hence $x = 120$

8. Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation satisfying $T(1,0) = (1,2,1)$ and $T(0,1) = (1,0,1)$. Find $T(1,-2)$.

A. $(0,2,0)$

B. $(0,0,0)$

$\boxed{\textbf{C.}}$ $(-1,2,-1)$

D. $(3,2,3)$

E. $(2,-1,3)$

F. $(1,0,3)$

Since $(1,-2) = (1,0) - 2(0,1)$ we have

$$T(1,-2) = T(1,0) - 2T(0,1) = (1,2,1) - 2(1,0,1)$$
$$= (1,2,1) - (2,0,2) = (-1,2,-1)$$

9. Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$, let I be the 2×2 identity matrix and let X be the 2×2-matrix satisfying

$$(AX + I)^T = 2I.$$

Then the first row of X is

A. $(-1, 1)$

B. $(1, 0)$

C. $(-2, 1)$

D. $(2, -1)$

E. $(3, -1)$

$\boxed{\textbf{F.}}$ $(3, -2)$

$$(AX + I)^T = 2I \Rightarrow AX + I = (2I)^T = 2I$$

$$\Rightarrow AX = I \Rightarrow X = A^{-1} = \frac{1}{\det A}\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$$

10. Let A, B and C be 2×2 matrices and assume $\det(A) = 2$, $\det(B) = 3$ and $\det(C) = 4$. Then the 2×2 matrix M satisfying $AMB = C$ has the properties

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A. $M = A^{-1}CB^{-1}$ and $\det(M) = 24$

B. $M = B^{-1}CA^{-1}$ and $\det(M) = 24$

$\boxed{\textbf{C.}}$ $M = A^{-1}CB^{-1}$ and $\det(M) = 2/3$

D. $M = B^{-1}CA^{-1}$ and $\det(M) = 2/3$

E. $M = CB^{-1}A^{-1}$ and $\det(M) = 24$

F. $M = CB^{-1}A^{-1}$ and $\det(M) = 2/3$

$$AMB = C \Rightarrow M = A^{-1}CB^{-1}$$

$$\det M = \det A^{-1} \det C \det B^{-1} = \frac{\det C}{\det A \cdot \det B} = \frac{4}{2 \cdot 3} = \frac{2}{3}$$

11. (10 pts) Let

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

(i) Find the rank of A and a basis of its row space.
(ii) Find a basis for the null space of A.

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[5] (i)

[3] $A \sim \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

$\Rightarrow$ rank $A = 3$, basis of row $A = \{(1\,1\,0\,1\,0), (0\,0\,1\,0\,1), (0\,0\,0\,1\,0)\}$

[5] (ii) null A = solutions of the linear system

$$\begin{aligned} x_1 + x_2 + x_4 &= 0 \\ x_3 + x_5 &= 0 \\ x_4 &= 0 \end{aligned} \quad , \text{ i.e. } \quad \begin{aligned} x_1 &= -x_2 \\ x_3 &= -x_5 \\ x_4 &= 0 \end{aligned}$$

[1] $\Rightarrow$ null $A = \left\{ \begin{pmatrix} -s \\ s \\ -t \\ 0 \\ t \end{pmatrix} ; s, t \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

basis of null A is $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

12. (10 pts) Let $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and let $U = \{A \in \mathbf{M}_{22}\,;\, AB = 0\}$.

[5] (i) Show that U is a subspace of the vector space $\mathbf{M}_{22}$ of 2×2-matrices.

[5] (ii) Find a basis of U.

(i) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $A \in U \Leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Leftrightarrow \begin{pmatrix} a+b & a+b \\ c+d & c+d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Leftrightarrow \begin{matrix} a+b=0 \\ c+d=0 \end{matrix} \Leftrightarrow \begin{matrix} b=-a \\ d=-c \end{matrix}$

Hence $U = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; \begin{matrix} a+b=0 \\ c+d=0 \end{matrix} \right\} = \left\{ \begin{pmatrix} a & -a \\ c & -c \end{pmatrix} \mid a, c \in \mathbb{R} \right\}$

$= \left\{ a\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \mid a, c \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \right\}$

Since U is the span of the two matrices $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$, it is a subspace

(ii) We check that the two matrices $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$ are linearly independent:

$s\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + t\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} s & -s \\ t & -t \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Leftrightarrow s = 0 = t$

Since $U = \operatorname{span}\left\{ \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \right\}$, a basis of U is $\left\{ \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \right\}$

<u>Note</u> One can also show (i) by verifying the 3 conditions of the subspace test:

(i) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 \in U$ since $0B = 0$

(ii) Suppose $A_1, A_2 \in U$, i.e. $A_1 B = 0 = A_2 B$. Then $(A_1 + A_2)B =$
$= A_1 B + A_2 B = 0 + 0 = 0$, hence $A_1 + A_2 \in U$

(iii) Suppose $A \in U$ and $s \in \mathbb{R}$. Then $(sA)B = s(AB) = s\,0 = 0$ using $AB = 0$. Hence $sA \in U$.

13. (12 pts) Consider the polynomials

$$p_1(x) = x + 1, \quad p_2(x) = x^2 - x + 1 \quad \text{and} \quad p_3(x) = x^2 + 1.$$

[5] (i) Determine if these polynomials are linearly independent.

[5] (ii) Does the polynomial x^2 lie in the span of $p_1(x)$, $p_2(x)$ and $p_3(x)$? If yes, write x^2 as a linear combination of these three polynomials.

[2] (iii) Is $\{p_1(x), p_2(x), p_3(x)\}$ a basis of the vector space $\mathbf{P}_2$ of polynomials of degree ≤ 2? Justify your answer.

(i) Assume $s_1 p_1(x) + s_2 p_2(x) + s_3 p_3(x) = 0$, i.e.

$$0 = s_1(x+1) + s_2(x^2 - x + 1) + s_3(x^2+1) =$$
$$= (s_1 + s_2 + s_3) + (s_1 - s_2)x + (s_2 + s_3)x^2$$

Then
$$\begin{aligned} s_1 + s_2 + s_2 &= 0 \\ s_1 - s_2 &= 0 \\ s_2 + s_3 &= 0 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} s_1 &= 0 \\ s_2 &= 0 \\ s_3 &= 0 \end{aligned}$$

Therefore, $\{p_1, p_2, p_3\}$ are linearly independent

(iii) Yes, since $\dim P_2 = 3$ and $\{p_1, p_2, p_3\}$ are linearly independent.

(ii) Since $P_2 = \operatorname{span}\{p_1, p_2, p_3\}$ and $x^2 \in P_2$ it is clear that

$$x^2 = s_1 p_1(x) + s_2 p_2(x) + s_3 p_3(x)$$

for certain $s_i \in \mathbb{R}$. We evaluate this condition

$$x^2 = s_1(x+1) + s_2(x^2 - x + 1) + s_3(x^2+1)$$
$$= (s_1 + s_2 + s_3) + (s_1 - s_2)x + (s_2 + s_3)x^2$$

which gives the linear system

$$\begin{aligned} s_1 + s_2 + s_3 &= 0 \\ s_1 - s_2 &= 0 \\ s_2 + s_3 &= 1 \end{aligned} \quad \Longleftrightarrow \quad \begin{cases} s_1 = s_2 \\ 2s_1 + s_3 = 0 \\ s_1 + s_3 = 1 \end{cases} \quad \Longleftrightarrow \quad \begin{aligned} s_1 &= s_2 \\ s_1 &= -1 \\ s_3 &= 2 \end{aligned}$$

Therefore

$$x^2 = -(x+1) - (x^2 - x + 1) + 2(x^2+1).$$

14. (8 pts) Give a basis of the space of solutions of the differential equation

$$f'' - 3f' + 2f = 0$$

and find the solution satisfying $f(0) = 1$ and $f'(0) = 0$.

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[3] The corresponding characteristic equation is $0 = x^2 - 3x + 2 = (x-2)(x-1)$, which has the two distinct roots 2 and 1. Therefore, a basis of the solution space is $\{e^{2x}, e^x\}$.

We want to find the solution f with $f(0)=1$ and $f'(0)=0$. We know $f(x) = c_1 e^{2x} + c_2 e^x$ for certain $c_i \in \mathbb{R}$, and

$$\begin{aligned} f(0) &= c_1 + c_2 = 1 \\ f'(0) &= 2c_1 + c_2 = 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 &= -1 \\ c_2 &= 2 \end{aligned}$$

Thus, $f(x) = 2e^x - e^{2x}$ is the unique solution of the differential equation with $f(0)=1$, $f'(0)=0$.

15. (10 pts) In each case, find a basis and the dimension of the subspace U of the vector space V. Justify your answers.

[5] (i) $V = \mathbb{R}^4$, $U = \{(a+b,\, a+c,\, b,\, c)\,;\, a,b,c \in \mathbb{R}\}$

[5] (ii) $V = \mathbf{P}_2$, $U = \{p(x) \in \mathbf{P}_2\,;\, p(0) + p(1) = 0\}$

(i) $U = \{ a(1,1,0,0) + b(1,0,1,0) + c(0,1,0,1) \mid a,b,c \in \mathbb{R} \}$

$= \text{span}\{(1100), (1010), (0101)\}$

We check that $\{(1100), (1010), (0101)\}$ is linearly independent

$0 = s_1(1100) + s_2(1010) + s_3(0101) = (s_1+s_2, s_1+s_3, s_2, s_3)$

$\Rightarrow \begin{array}{l} s_1+s_2 = 0 \\ s_1+s_3 = 0 \\ s_2 = 0 \\ s_3 = 0 \end{array} \Rightarrow s_1 = s_2 = s_3 = 0$, i.e. the 3 vectors are linearly independent.

Hence $\{(1100), (1010), (0101)\}$ is a basis of U.

(ii) Let $p(x) = a + bx + cx^2 \in P_2$. Then $p \in U \Leftrightarrow p(0) + p(1) = 0$

$\Leftrightarrow a + (a+b+c) = 0 \Leftrightarrow 2a+b+c = 0 \Leftrightarrow c = -2a - b$

It follows that $U = \{ a + bx - (2a+b)x^2 \mid a,b \in \mathbb{R} \} =$

$= \{ a(1-2x^2) + b(x-x^2) \mid a,b \in \mathbb{R} \} = \text{span}\{1-2x^2, x-x^2\}$.

We check that $1-2x^2$ and $x-x^2$ are linearly independent.

$0 = a(1-2x^2) + b(x-x^2) = a + bx - (2a+b)x^2 \Rightarrow a = 0 = b$

This means that $1-2x^2$ and $x-x^2$ are linearly independent. Therefore $\{1-2x^2, x-x^2\}$ is a basis of U.

16. (10 pts) The eigenvalues of the matrix $A = \begin{pmatrix} -8 & 6 \\ -9 & 7 \end{pmatrix}$ are -2 and 1.

(i) Find a basis of each eigenspace of A.

(ii) Is A diagonalizable? Justify your answer. If your answer is yes, find a matrix P such that $P^{-1}AP$ is diagonal and give $P^{-1}AP$.

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[6] (i) $E_{-2}(A) = \text{null}(-2I_2 - A)$ where

$-2I_2 - A = \begin{pmatrix} 6 & -6 \\ 9 & -9 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$. Hence $x - y = 0$, i.e. $x = y$

$E_{-2}(A) = \{\begin{pmatrix} x \\ x \end{pmatrix} \mid x \in \mathbb{R}\} = \{x\begin{pmatrix} 1 \\ 1 \end{pmatrix} \mid x \in \mathbb{R}\} = \text{span}\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$

Since $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \neq 0$, it is linearly independent. Hence $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a basis of $E_{-2}(A)$.

$E_1(A) = \text{null}(I - A) = \begin{pmatrix} 9 & -6 \\ 9 & -6 \end{pmatrix} \sim \begin{pmatrix} 3 & -2 \\ 0 & 0 \end{pmatrix}$, ie $3x = 2y$, so

$E_1(A) = \{t\begin{pmatrix} 2 \\ 3 \end{pmatrix} \mid t \in \mathbb{R}\} = \text{span}\{\begin{pmatrix} 2 \\ 3 \end{pmatrix}\}$, and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is a basis of $E_1(A)$.

[4] (ii) Yes, because $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ are linearly independent since they are
[1] eigenvectors for different eigenvalues. Since $\dim \mathbb{R}^2 = 2$, it follows that $\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}\}$ is a basis of $\mathbb{R}^2$ consisting of eigenvectors of A. The matrix

[2] $$P = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$$

is invertible, and $P^{-1}AP = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$.