1 For what value(s) of k are the vectors $(1,-2,1)$, $(2,-1,4)$, $(-1,8,k)$ linearly independent?

A. all $k \neq 2$ **B.** all $k \neq 3$ **C.** all $k \neq -1$
D. $k=-1$ only **E.** $k=3$ only **F.** $k=2$ only

2 Find the main diagonal of the inverse of $A = \begin{pmatrix} 1 & -2 & -3 \\ -2 & 2 & 4 \\ -3 & 0 & 2 \end{pmatrix}$.

A. $(-1, -7/2, 3)$ **B.** $(2, 1, -7/2)$ **C.** $(2, -7/2, -1)$
D. $(7/2, 2, -1)$ **E.** $(2, 1, -1)$ **F.** $(5/2, 7/2, 3/2)$

3. If $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is a linear transformation and we are given that $T(1,0,0) = (1,2)$ and $T(0,0,1) = (2,3)$, then $T(5,0,2) =$

A. $(9,16)$
B. $(9,-2)$
C. $(1,1)$
D. $T(5,0,2)$ cannot be calculated
E. $(-7,12)$
F. $(3,5)$

14. Find the 2×2 matrix A which satisfies the equation

$$(2I_2 - A)^{-1} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$$

and give the sum of all of its entries.

A. -3 **B.** 0 **C.** 4 **D.** -6 **E.** 1 **F.** 7

5. Find numbers a, b and c for which

$$\begin{pmatrix} 2 & -1 & 1 \\ 1 & -1 & -1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix},$$

and give the value of b.

A. There is no solution
B. $b = 0$
C. $b = -2$
D. $b = 2$
E. $b = 1$
F. $b = -1$

6. Find all points of intersection of the line $(x, y, z) = (-4, 3, 4) + t(1, 1, 1)$ and the plane $3x - y + 2z = 1$.

A. $(-3, 4, 5)$
B. The line lies in the plane.
C. $(-2, 5, 6)$
D. $(1, 4, 1)$
E. The line does not intersect the plane.
F. $(0, -7, -3)$

7. The coefficient matrix of a certain homogeneous linear system of 10 linear equations in 16 variables has rank 7. How many parameters will there be in the solution?

A. 16 B. 6 C. 7 D. none E. 9 F. 3

8. Which of the following statements are **true** ?

(1) The set $\{ (x, y, 0) \in \mathbb{R}^3 \; ; \; x = y^2 \}$ is a subspace of $\mathbb{R}^3$.

(2) The set $\{ x + ax^3 \; ; \; a \in \mathbb{R} \}$ is a subspace of $\mathbf{P}_3$.

(3) The set $\{ (a, b, c) \in \mathbb{R}^3 \; ; \; 2a - c = 0 \}$ is a subspace of $\mathbb{R}^3$.

(4) The set $\left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \; ; \; a, b, c \in \mathbb{R} \right\}$ is a subspace of M_{22}.

A. 2 and 3 **B.** 2, 3 and 4 **C.** 1, 2 and 3
D. 3 and 4 **E.** 4 **F.** 1 and 2

9. [14 points]

(a) Indicate whether each of the following statements is **True** or **False**. Each correct answer is worth **2** points; a wrong answer is worth **-1** point; no answer is worth 0 points.

____ $\{x^3 - x,\ 2x^3 + 5x^2 - 2x,\ 2x^2\}$ is a linearly independent subset of $\mathbf{P}_3$.

____ $\dim \mathbf{P}_n = n$.

____ There is a set of 4 vectors whose span is $\mathbb{R}^7$.

Answer each of the following in the blank provided:

(b) $(AB)^T =$ ____________ for all 4×4 matrices A and B.

$\det(3A) =$ ________________ for every 5×5 matrix A.

(c) State two equivalent conditions for a square matrix A to be invertible:

Condition 1:

Condition 2:

10. [6 points] Answer each of the following in the blank provided:

(a) A is a 4×5 matrix and B is a 5×3 matrix. If C is a *square* matrix and $(AB)^T C$ is defined, what is the size of C? ________

(b) Give an example of a non-zero 2×2 matrix which is symmetric, i.e., equal to its transpose, but not invertible.

____

(c) Find a complex number x for which $\begin{vmatrix} x-4 & 3 \\ -1 & x-2 \end{vmatrix} = 1$. Answer: $x =$ ________

11. [10 points] Determine for which value(s) of b the system

$$\begin{aligned}(b+2)\,x + \qquad 3\,y &= b+1 \\ x + (2-b)\,y &= 0\end{aligned}$$

has

(a) no solution
(b) infinitely many solutions
(c) a unique solution

12. [8 points] Suppose u, v, w are linearly independent vectors of some vector space V.
   (a) Show that $u + w$, $u - w$, $u + v + w$ are also linearly independent.
   (b) If $\{u, v, w\}$ is a basis of V, why does the result of (a) allow you to conclude that $\{u + w,\ u - w,\ u + v + w\}$ is also a basis of V?

13. [10 points] Let $U = \{ (a, b, c, d) \in \mathbb{R}^4 \; ; \; b = c - d \}$.

(a) Show that U is a subspace of $\mathbf{R}^4$.

(b) Find a basis of U, and give the value of $\dim U$. Justify your answer.

14. [12 points] The characteristic polynomial of the matrix

$$A = \begin{pmatrix} 3 & 2 & -1 \\ -1 & 0 & 1 \\ 2 & 4 & 0 \end{pmatrix}$$

is $c_A(x) = (x+1)(x-2)^2$.

(i) Find all eigenvalues of A and a basis of each eigenspace.

(ii) Decide if A is diagonalizable (justify your answer). If so, give a matrix P such that $P^{-1}AP$ is diagonal and write down $P^{-1}AP$.