MAT 1341 D **FINAL EXAM** **April 26, 2000**

Instructor: Erhard Neher **Time. 180 minutes**

Family Name: ______________________

First Name: ______________________

Student number __________

Closed book test. No notes allowed.

Non-programmable calculators with single-line display may be used.

The test consists of 14 questions. The first 8 questions are multiple choice, and are worth 5 points each. Questions 9 and 10 consists of several short-answer questions, while the remaining 4 questions are written questions. The value of each of these questions is indicated at the start of the question. The total number of points for the test is 100.

Do your work for each question in the space provided. If you need additional space, use the back of the preceding page.

Record the LETTER corresponding to your answer for each multiple choice question in the appropriate box below.

1.	2.	3.	4.	5.	6.	7.	8.

Questions 11 through 14 require a worked out answer. Write your answers clearly and logically. Partial marks may be earned for partial solutions of these questions. To obtain full marks your answers should be complete and clearly explained.

1. For what value(s) of k are the vectors $(1,-2,1)$, $(2,-1,4)$, $(-1,8,k)$ linearly independent?

[**A.**] all $k \neq 3$ **B.** all $k \neq 2$ **C.** all $k \neq -1$
D. $k = 3$ only **E.** $k = 2$ only **F.** $k = -1$ only

Suppose $x(1,-2,1) + y(2,-1,4) + z(-1,8,k) = 0$

Equivalently $(x+2y-z, -2x-y+8z, x+4y+zk) = (0,0,0)$, or

$$\begin{aligned} x+2y-z &= 0 \\ -2x-y+8z &= 0 \\ x+4y+kz &= 0 \end{aligned}$$

We need to find the condition on k such that this homogeneous linear system has only the trivial solution. Corresponding coefficient matrix:

$$A = \begin{pmatrix} 1 & 2 & -1 \\ -2 & -1 & 8 \\ 1 & 4 & k \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 6 \\ 0 & 2 & k+1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & k+1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & k-3 \end{pmatrix}$$

only trivial solution $\iff$ rank $A = 3$ $\iff$ $k-3 \neq 0$ $\iff$ $k \neq 3$

2. Find the main diagonal of the inverse of $A = \begin{pmatrix} 1 & -2 & -3 \\ -2 & 2 & 4 \\ -3 & 0 & 2 \end{pmatrix}$.

A. (2, −7/2, −1) **B.** (5/2, 7/2, 3/2) **C.** (2, 1, −1)
D. (−1, −7/2, 3) **E.** (7/2, 2, −1) **F.** (2, 1, −7/2)

We use the inversion algorithm to find A^{-1}:

$$[A \mid I_3] = \left(\begin{array}{ccc|ccc} 1 & -2 & -3 & 1 & 0 & 0 \\ -2 & 2 & 4 & 0 & 1 & 0 \\ -3 & 0 & 2 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -2 & -3 & 1 & 0 & 0 \\ 0 & -2 & -2 & 2 & 1 & 0 \\ 0 & -6 & -7 & 3 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -2 & -3 & 1 & 0 & 0 \\ 0 & 2 & 2 & -2 & -1 & 0 \\ 0 & 6 & 7 & -3 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -2 & -3 & 1 & 0 & 0 \\ 0 & 2 & 2 & -2 & -1 & 0 \\ 0 & 0 & 1 & 3 & 3 & -1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & -1 & 0 \\ 0 & 2 & 2 & -2 & -1 & 0 \\ 0 & 0 & 1 & 3 & 3 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 2 & -1 \\ 0 & 2 & 0 & -8 & -7 & 2 \\ 0 & 0 & 1 & 3 & 3 & -1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 2 & -1 \\ 0 & 1 & 0 & -4 & -7/2 & 1 \\ 0 & 0 & 1 & 3 & 3 & -1 \end{array}\right), \text{ so } A^{-1} = \begin{pmatrix} 2 & 2 & -1 \\ -4 & -7/2 & 1 \\ 3 & 3 & -1 \end{pmatrix}$$

3. If $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is a linear transformation and we are given that $T(1,0,0) = (1,2)$ and $T(0,0,1) = (2,3)$, then $T(5,0,2) =$

A. $(9,16)$ (boxed)
B. $T(5,0,2)$ cannot be calculated
C. $(3,5)$
D. $(-7,12)$
E. $(1,1)$
F. $(9,-2)$

Observe $(5,0,2) = 5(1,0,0) + 2(0,0,1)$. Since T is linear, we get

$$T(5,0,2) = T(5(1,0,0) + 2(0,0,1))$$
$$= 5T(1,0,0) + 2T(0,0,1)$$
$$= 5(1,2) + 2(2,3) =$$
$$= (5,10) + (4,6) = (9,16)$$

4. Find the 2×2 matrix A which satisfies the equation

$$(2I_2 - A)^{-1} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$$

and give the sum of all of its entries.

[A.] 4 **B.** 7 **C.** 1 **D.** 0 **E.** -6 **F.** -3

$$(2I_2 - A)^{-1} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \Leftrightarrow$$

$$\Leftrightarrow \quad 2I_2 - A = \left((2I_2 - A)^{-1}\right)^{-1} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}^{-1} = -\begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}$$

$$\Leftrightarrow \quad A = 2I_2 - \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix} =$$

$$= \begin{pmatrix} 2+3 & -2 \\ -2 & 2+1 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ -2 & 3 \end{pmatrix}$$

Sum of all entries of A is $5 - 2 - 2 + 3 = 4$

5. Find numbers a, b and c for which

$$\begin{pmatrix} 2 & -1 & 1 \\ 1 & -1 & -1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix},$$

and give the value of b.

A. $b = 1$	**B.** $b = 0$	**C.** $b = 2$
D. $b = -1$	**E.** $b = -2$	**F.** There is no solution

To solve is :

$$2a - b + c = 0$$
$$a - b - c = -2$$
$$2b + c = 3$$

; in matrix form :

$$\begin{pmatrix} 2 & -1 & 1 & 0 \\ 1 & -1 & -1 & -2 \\ 0 & 2 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -1 & -2 \\ 2 & -1 & 1 & 0 \\ 0 & 2 & 1 & 3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & -5 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

ie $a = 0$, $b = 1$, $c = 1$

6. Find all points of intersection of the line $(-4,3,4)+t(1,1,1)$ and the plane $3x-y+2z=1$.

[A] $(-2,5,6)$
B. The line lies in the plane.
C. The line does not intersect the plane.
D. $(-3,4,5)$
E. $(1,4,1)$
F. $(0,-7,-3)$

A point of the line $(x,y,z)=(-4,3,4)+t(1,1,1)=(-4+t,3+t,4+t)$

lies on the plane $3x-y+2z=1 \iff$

$$1 = 3(-4+t)-(3+t)+2(4+t)$$
$$= -12+3t-3-t+8+2t = 4t-7 \iff 4t=7+1$$
$$\iff t=2$$

Hence the point of intersection is

$$(-4,3,4)+2(1,1,1) = (-4+2,3+2,4+2) = (-2,5,6)$$

7. The coefficient matrix of a certain homogeneous linear system of 10 linear equations in 16 variables has rank 7. How many parameters will there be in the solution?

[A.] 9 **B.** 6 C. 7 **D.** none E. 3 **F.** 16

# of parameters in the solution

= # of variables − rank of coefficient matrix

= 16 − 7 = 9

8. Which of the following statements are **true** ?

(1) The set $\{ (x, y, 0) \in \mathbb{R}^3 \ ; \ x = y^2 \}$ is a subspace of $\mathbb{R}^3$.

(2) The set $\{ x + ax^3 \ ; \ a \in \mathbb{R} \}$ is a subspace of $\mathbf{P}_3$.

(3) The set $\{ (a, b, c) \in \mathbb{R}^3 \ ; \ 2a - c = 0 \}$ is a subsapce of $\mathbb{R}^3$.

(4) The set $\left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \ ; \ a, b, c \in \mathbb{R} \right\}$ is a subspace of M_{22}.

[A] 3 and 4 **B.** 2 and 3 **C.** 1 and 2
D. 4 **E.** 1, 2 and 3 **F.** 2, 3 and 4

(1) False, since $U = \{(x, y, 0) \in \mathbb{R}^3 \ ; \ x = y^2\}$ is not closed under scalar multiples: $(1, 1, 0) \in U$ but $(-1)(1,1,0) = (-1,-1,0) \notin U$.

(2) False, since the zero polynomial $0 + 0x + 0x^2 + 0x^3$ is not in the set

(3) True, since $\{(a,b,c) \in \mathbb{R}^3 \ ; \ 2a - c = 0\} = \{(a, b, 2a) \in \mathbb{R}^3 ; a, b \in \mathbb{R}\}$
$= \{a(1,0,2) + b(0,1,0) \ ; \ a, b \in \mathbb{R}\} = \text{span}\{(1,0,2), (0,1,0)\}$

(4) True, since $\left\{\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} ; a, b, c \in \mathbb{R}\right\} =$
$= \left\{ a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \ ; \ a, b, c \in \mathbb{R} \right\}$
$= \text{span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

9. [14 points]

(a) Indicate whether each of the following statements is **T**rue or **F**alse. Each correct answer is worth 2 points; a wrong answer is worth -1 point; no answer is worth 0 points.

F $\{x^3 - x,\ 2x^3 + 5x^2 - 2x,\ 2x^2\}$ is a linearly independent subset of $\mathbf{P}_3$.

F $\dim \mathbf{P}_n = n$. ($\dim P_n = n+1$)

T There is a set of 4 vectors whose span is $\mathbb{R}^3$.
(Example: (1,0,0), (0,1,0), (0,0,1), (1,1,0) span $\mathbb{R}^3$)

Answer each of the following in the blank provided:

(b) $(AB)^T =$ $B^T A^T$ for all 4×4 matrices A and B.

$\det(3A) =$ $3^5 \det(A)$ for every 5×5 matrix A.

(c) State two equivalent conditions for a square matrix A to be invertible:
(I assume that A is an n×n matrix)

Condition 1: rank $A = n$

Condition 2: $\det A \neq 0$

other possible conditions:

- the reduced row echelon form of A is I_n
- the rows of A span $\mathbb{R}^n$
- the rows of A are a basis of $\mathbb{R}^n$
- the rows of A are linearly independent
- the columns of A span $\mathbb{R}^n$
- the columns of A are a basis of $\mathbb{R}^n$
- the columns of A are linearly independent
- the homogeneous linear system $AX=0$ has only the trivial solution

10. [6 points] Answer each of the following in the blank provided:

(a) A is a 4×5 matrix and B is a 5×3 matrix. If C is a *square* matrix and $(AB)^T C$ is defined, what is the size of C? 4×4

AB is 4×3, so $(AB)^T$ is 3×4, hence C is 4×4

(b) Give an example of a non-zero 2×2 matrix which is symmetric, i.e., equal to its transpose, but not invertible.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

(c) Find a complex number x for which $\begin{vmatrix} x-4 & 3 \\ -1 & x-2 \end{vmatrix} = 1$. Answer: $x =$ $3 \pm i$

$$1 = (x-4)(x-2) + 3 = x^2 - 6x + 11 \iff x^2 - 6x + 10 = 0$$

$$\iff (x-3)^2 = -1 \iff x - 3 = \pm i \iff x = 3 \pm i$$

11. [10 points] Determine for which value(s) of b the system

$$\begin{aligned}(b+2)\,x + \quad 3\,y &= b+1\\ x + (2-b)\,y &= 0\end{aligned}$$

has

(a) no solution
(b) infinitely many solutions
(c) a unique solution

In matrix form

$$\left(\begin{array}{cc|c} (b+2) & 3 & b+1 \\ 1 & 2-b & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 2-b & 0 \\ b+2 & 3 & b+1 \end{array}\right) \sim \left(\begin{array}{ccc} 1 & 2-b & 0 \\ 0 & a & b+1 \end{array}\right)$$

where $a = 3-(b+2)(2-b) = 3-(4-b^2) = b^2-1$

(a) no solution $\Leftrightarrow$ $a=0,\ b+1\neq 0$ $\Leftrightarrow$ $b=\pm 1,\ b+1\neq 0$

$\Leftrightarrow$ $b=1$

(b) infinitely many solutions $\Leftrightarrow$ $a=0=b+1$ $\Leftrightarrow$ $b=-1$

(c) a unique solution $\Leftrightarrow$ $a\neq 0$ $\Leftrightarrow$ $b\neq \pm 1$.

12. [8 points] Suppose u, v, w are linearly independent vectors of some vector space V.

(a) Show that $u+w$, $u-w$, $u+v+w$ are also linearly independent.

(b) If $\{u, v, w\}$ is a basis of V, why does the result of (a) allow you to conclude that $\{u+w, u-w, u+v+w\}$ is also a basis of V?

(a) Suppose $a(u+w)+b(u-w)+c(u+v+w)=0$

We must show $a=b=c=0$.

The condition means

$$0 = au + aw + bu - bw + cu + cv + cw$$
$$= (a+b+c)u + cv + (a-b+c)w$$

Since u, v, w are linearly independent, we obtain

$$a+b+c=0$$
$$c=0$$
$$a-b+c=0$$

The only solution of this system is $a=b=c=0$.

(b) Since $\{u, v, w\}$ is a basis of V, we have $\dim V = 3$.

By a result in §5.4, every set of 3 linearly independent vectors is a basis of V. But by (a) we know that $u+w$, $u-w$, $u+v+w$ are linearly independent. So, $\{u+w, u-w, u+v+w\}$ is a basis of V.

13. [10 points] Let $U = \{ (a,b,c,d) \in \mathbb{R}^4 \; ; \; b = c - d \}$.

(a) Show that U is a subspace of $\mathbf{R}^4$.

(b) Find a basis of U, and give the value of $\dim U$. Justify your answer.

(a) $U = \{(a, c-d, c, d) \in \mathbb{R}^4 \; ; \; a, c, d \in \mathbb{R}\}$

$= \{a(1,0,0,0) + c(0,1,1,0) + d(0,-1,0,1) \; ; \; a,c,d \in \mathbb{R}\}$

$= \text{span}\{(1,0,0,0), (0,1,1,0), (0,-1,0,1)\}$, so U is a subspace

2nd solution: check subspace test:

- $(0,0,0,0) \in U$ since $0 = 0 - 0$
- Let $(a,b,c,d) \in U$ and $(a_1, b_1, c_1, d_1) \in U$, i.e. $b = c-d$ and $b_1 = c_1 - d_1$. Then $(a,b,c,d) + (a_1,b_1,c_1,d_1) = (a+a_1, b+b_1, c+c_1, d+d_1) \in U$ since $b+b_1 = (c+c_1) - (d+d_1)$
- Let $(a,b,c,d) \in U$, i.e., $b = c-d$, and let $r \in \mathbb{R}$. Then $r(a,b,c,d) = (ra, rb, rc, rd) \in U$ since $rb = r(c-d) = rc - rd$.

(b) By the solution of (a), we know that U is spanned by the 3 vectors $(1,0,0,0), (0,1,1,0), (0,-1,0,1)$. We show that they are linearly independent:

$0 = x(1,0,0,0) + y(0,1,1,0) + z(0,-1,0,1)$

$= (x, y-z, y, z) \Rightarrow x = 0 = y = z$.

Hence, the 3 vectors form a basis of U, and $\dim U = 3$.

14. [12 points] The characteristic polynomial of the matrix

$$A = \begin{pmatrix} 3 & 2 & -1 \\ -1 & 0 & 1 \\ 2 & 4 & 0 \end{pmatrix}$$

is $c_A(x) = (x+1)(x-2)^2$.

(i) Find all eigenvalues of A and a basis of each eigenspace.

(ii) Decide if A is diagonalizable (justify your answer). If so, give a matrix P such that $P^{-1}AP$ is diagonal and write down $P^{-1}AP$.

(i) The eigenvalues of A are the roots of the characteristic polynomial. Therefore, we have the two eigenvalues -1 and 2.

$E_{-1}(A) = \text{null}\,(-I_3 - A)$:

$$-I_3 - A = \begin{pmatrix} -4 & -2 & 1 \\ 1 & -1 & -1 \\ -2 & -4 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -1 \\ -2 & -4 & -1 \\ -4 & -2 & +1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -1 \\ 0 & -6 & -3 \\ 0 & -6 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

hence $x = -y = \frac{1}{2}t$, $y = -\frac{1}{2}t$, $z = t$, and $E_{-1}(A) = \mathbb{R}\begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix} = \mathbb{R}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ has dimension 1

with basis $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ (for example)

$E_2(A) = \text{null}\,(2I_3 - A)$

$$2I_3 - A = \begin{pmatrix} -1 & -2 & 1 \\ 1 & 2 & -1 \\ -2 & -4 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ has rank 1

Solution $x = z - 2y$, $\dim E_2(A) = 3 - \text{rank}(2I_3 - A) = 2$

basis vectors $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

(ii) For each eigenvalue, we have $\dim E_\lambda(A) =$ multiplicity of λ in $c_A(x)$

So A is diagonalizable. For

$$P = \begin{pmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$ we obtain $$P^{-1}AP = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$