1. Let V be a vector space of dimension n. Which of the following statements are true?

(1) Every spanning set of V can be extended to a basis of V.
(2) Every set of n vectors in V is linearly independent.
(3) Every set of n linearly independent vectors in V is a basis of V.

A. (1) and (2)
B. (1) only
C. all of them
D. none of them
E. (2) only
F. (1) and (3)
G. (3) only
H. (2) and (3)

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2. Determine the number of parameters in the general solution of the system

$$\begin{cases} x_1 + x_2 - x_3 - x_4 = 1 \\ x_1 + x_2 + 2x_4 = 3 \\ 2x_1 + 2x_2 - x_3 + x_4 = 0 \end{cases}$$

A. 0 parameters
B. 4 parameters
C. The system has no solution.
D. 3 parameters
E. 2 parameters
F. 1 parameter

3. Consider the matrix $A = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

Let $r = \text{rank}(A)$ and let d be the dimension of the solution space of the homogeneous system $AX = 0$. Determine the values of r and d.

A. $r = 2$ and $d = 0$ **B.** $r = 3$ and $d = 1$ **C.** $r = 3$ and $d = 2$
D. $r = 2$ and $d = 3$ **E.** $r = 3$ and $d = 3$ **F.** $r = 3$ and $d = 5$

4). Given that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis of V, which of the following are also bases for V?

(1) $\{\mathbf{u}+\mathbf{v}-\mathbf{w}, \mathbf{v}+\mathbf{w}\}$
(2) $\{\mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{w}, \mathbf{u}-\mathbf{v}, \mathbf{v}+\mathbf{w}\}$
(3) $\{\mathbf{u}+\mathbf{v}+\mathbf{w}, \mathbf{u}+\mathbf{w}, \mathbf{v}\}$
(4) $\{\mathbf{u}, \mathbf{v}-\mathbf{w}, \mathbf{v}+\mathbf{w}\}$

A. (2) and (3) **B.** (1) and (3) **C.** (1) and (4)
D. (3) only **E.** (4) only **F.** (2), (3) and (4)

5). Find the roots of the following quadratic equation:

$$2x^2 + 4x + 3 = 0.$$

A. $x = -1 + \frac{\sqrt{2}}{2}i$
B. $x = \frac{-3 \pm \sqrt{14}\,i}{4}$
C. $x = -2 \pm \sqrt{2}\,i$
D. $x = \frac{3 \pm \sqrt{3}\,i}{4}$
E. $x = 2 \pm 2\sqrt{2}\,i$
F. $x = 1 \pm \frac{\sqrt{14}}{2}\,i$

6). At age 38, the age of Mr. Green is the sum of the ages of his three daughters. The age of the oldest daughter equals the sum of the ages of her two sisters. Also, the age difference between the oldest and youngest sisters is 14 years. What is the age of the youngest of the three daughters?

A. 6 years
B. 5 years
C. 3 years
D. 2 years
E. Cannot be determined from the given data.
F. 4 years

7) For what value of k do the following two lines intersect?

$$\begin{aligned} x &= -3 + r \\ y &= 3 + 2r \\ z &= -r \end{aligned} \qquad \begin{aligned} x &= 2 - 2t \\ y &= 10 - 2t \\ z &= 1 + kt \end{aligned}$$

A. $k = 4$
B. $k = 6$
C. $k = 0$
D. $k = -2$
E. $k = 3$
F. $k = -1$

8. Let $A = (a_{ij})$ be a 4×3 matrix and $B = (b_{ij})$ be a 5×3 matrix. Give a formula for the $(3,2)$-element of AB^T.

A. $(AB^T)_{32} = \sum_{k=1}^{5} a_{2k}b_{3k}$

B. $(AB^T)_{32} = \sum_{k=1}^{3} a_{2k}b_{3k}$

C. $(AB^T)_{32} = \sum_{k=1}^{5} a_{3k}b_{2k}$

D. $(AB^T)_{32} = \sum_{k=1}^{3} a_{2k}b_{k3}$

E. $(AB^T)_{32} = \sum_{k=1}^{3} a_{3k}b_{k2}$

F. $(AB^T)_{32} = \sum_{k=1}^{3} a_{3k}b_{2k}$

9. Use **Cramer's Rule** to solve for x_2 in the following linear system:

$$\begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 3 \\ 5 & 3 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix}$$

A. $x_2 = -10/3$ **B.** $x_2 = -14/9$ **C.** $x_2 = -25/6$
D. $x_2 = 10/3$ **E.** $x_2 = 14/9$ **F.** $x_2 = 25/6$

10. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and let B be any 3×4 matrix. What can one say about row 2 of the product AB?

A. It is equal to row 1 of B multiplied by -2.
B. It is equal to row 2 of B multiplied by -2.
C. It is equal to row 1 of B minus two times row 2 of B.
D. It is equal to row 2 of B minus two times row 1 of B.
E. It is equal to row 2 of B.
F. It is equal to row 3 of B minus two times row 2 of B.

11. [14 points]

(a) Indicate whether each of the following statements is **True** or **False**. Each correct answer is worth 2 points; a wrong answer is worth -1 point; no answer is worth 0 points.

____ If the reduced row echelon (RREF) form of a matrix A has a row of 0's, then the linear system $AX = 0$ will have infinitely many solutions.

____ There exist square matrices A and B for which $AB = BA$.

____ If A is an invertible matrix, its columns are linearly dependent.

____ If A and B are invertible $n \times n$ matrices, then $(AB)^{-1} = A^{-1}B^{-1}$.

Complete each of the following statements:

(b) If $A = \begin{pmatrix} 3 & -2 \\ -5 & 4 \end{pmatrix}$ then $A^{-1} = \begin{pmatrix} & \\ & \end{pmatrix}$

(c) Let $P = (2, 1, 4)$ and $Q = (6, 3, 1)$. The point $\frac{1}{3}$ the way from P to Q is $(\quad , \quad , \quad)$

(d) If A is a 3×3 matrix and $\det A = 4$, then $\det(2A^{-1}) =$ ____

12. [6 points] Answer each of the following in the blank provided:

(a) Suppose $T : \mathbf{P}_2 \to \mathbf{P}_3$ is a linear map satisfying

$$T(1+x^2) = 1 - x, \quad T(x + 2x^2) = 1 + x^3, \quad T(x^2) = 1 + x^3.$$

Find $T(x)$.

Answer: $T(x)$ = ________

(b) Let A be an $n \times n$ matrix. Prove that $A + A^T$ is symmetric.

(c) Give the definition of the *span* of a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ in a vector space V.

13. [10 points] Determine the values of b and c for which the system

$$\begin{cases} x + 2y + z = b \\ x + \ y + (1-b)z = b - c \\ x + \ y + z = 1 - 2c \end{cases}$$

has

(a) exactly one solution
(b) no solution
(c) infinitely many solutions

In the case where there are infinitely many solutions, give the general solution.

14. [10 points] Consider the subspace U of M_{22} given by $U = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} ; a, b, c \in \mathbf{R} \right\}$ and the matrices $\mathbf{v}_1 = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$ and $\mathbf{v}_3 = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$. Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a basis for U? Either prove that it is, or show that it isn't. Justify your answer.

15. [10 points] (a) Show that $U = \{ p(x) \in \mathbf{P}_3 \,;\, 2p(0) = p(2) \}$ is a subspace of $\mathbf{P}_3$.

(b) Find a basis of U, and determine $\dim U$.

16. [10 points] (a) Find all eigenvalues of the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Is A diagonalizable? Please justify your answer.

(b) The eigenvalues of the matrix

$$B = \begin{pmatrix} -5 & 0 & 6 \\ -3 & 1 & 3 \\ -3 & 0 & 4 \end{pmatrix}$$

are $\lambda_1 = 1$ and $\lambda_2 = -2$. Find all eigenspaces of B. Is B diagonalizable? Please justify your answer.

[more work space on next page...]