

1. Let  $V$  be a vector space of dimension  $n$ . Which of the following statements are true?

- (1) Every spanning set of  $V$  can be extended to a basis of  $V$ .  
 (2) Every set of  $n$  vectors in  $V$  is linearly independent.  
 (3) Every set of  $n$  linearly independent vectors in  $V$  is a basis of  $V$ .
- A. (1) and (2)                      B. (2) only                      C. (1) only  
 D. (1) and (3)                      E. (3) only                      F. none of them  
 G. all of them                      H. (2) and (3)

(1) False: a spanning set need not be linearly independent  
 (Example:  $v_1 = (1, 0)$ ,  $v_2 = (0, 1)$ ,  $v_3 = (1, 1)$  is a spanning set of  $\mathbb{R}^2$  which is not linearly independent). Adding vectors will not change linear dependence.

The correct statement would have been: "Every linearly independent set" can be extended to a basis" or "Every spanning set contains a basis".

(2) False: For example,  $v_1 = (1, 0)$  and  $v_2 = (2, 0)$  are 2 vectors in  $\mathbb{R}^2$  which are linearly dependent. The correct statement is: "Every spanning set of  $n$  vectors in  $V$  is linearly independent."

(3) True: § 5.4 Thm. 3, p. 222

2. Determine the number of parameters in the general solution of the system

$$\begin{cases} x_1 + x_2 - x_3 - x_4 = 1 \\ x_1 + x_2 + 2x_4 = 3 \\ 2x_1 + 2x_2 - x_3 + x_4 = 0 \end{cases}$$

- A. 1 parameter  
C. 4 parameters  
E. 2 parameters

- B. 3 parameters  
D. 0 parameters  
**E.** The system has no solution.

associated  
augmented  
matrix

$$\begin{pmatrix} 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 2 & 3 \\ 2 & 2 & -1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 3 & -2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Because the last row is  $(0 \ 0 \ 0 \ 0 \ 1)$ , the system has no solution.

3. Consider the matrix  $A = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ .

Let  $r = \text{rank}(A)$  and let  $d$  be the dimension of the solution space of the homogeneous system  $AX = 0$ . Determine the values of  $r$  and  $d$ .

A.  $r = 3$  et  $d = 3$

B.  $r = 3$  et  $d = 5$

C.  $r = 2$  et  $d = 0$

D.  $r = 3$  et  $d = 2$

E.  $r = 2$  et  $d = 3$

F.  $r = 3$  et  $d = 1$

The matrix is already in ref (row echelon form), its rank is  $r=3$ . The corresponding homogeneous system has  $n=5$  variables, hence  $d = n - r = 5 - 3 = 2$

4. Given that  $\{u, v, w\}$  is a basis of  $V$ , which of the following are also bases for  $V$ ?

- (1)  $\{u+v+w, v+w\}$   
 (2)  $\{u+v, u+w, u-v, v+w\}$   
 (3)  $\{u+v+w, u+w, v\}$   
 (4)  $\{u, v-w, v+w\}$

- A. (1) and (3)                      B. (3) only                      C. (2) and (3)  
 D. (2), (3) and (4)                E. (4) only                      F. (1) and (4)

$\{u, v, w\}$  basis of  $V \Rightarrow \dim V = 3 \Rightarrow$  every basis of  $V$  has exactly 3 vectors.

(1) not a basis, because only 2 vectors

(2) not a basis, because 4 vectors

(3) not a basis since  $v_1 = u+v+w = -(u+w) + v = v_2 + v_3$ ,  
 so the set is not linearly independent

(4) is a basis:

$\{u, v-w, v+w\}$  is linearly independent: suppose  
 $0 = x u + y(v-w) + z(v+w) = x u + (y+z)v + (z-y)w$   
 $\Rightarrow$   $x=0 = y+z = z-y \rightarrow x=y=z=0$ .

Because  $\dim V = 3$ , every set of 3 lin. indep. vectors is a basis

5. Find the roots of the following quadratic equation:

$$2x^2 + 4x + 3 = 0.$$

A.  $x = -2 \pm \sqrt{2}i$

B.  $x = \frac{3 \pm \sqrt{3}i}{4}$

C.  $x = 1 \pm \frac{\sqrt{14}}{2}i$

D.  $x = -1 \pm \frac{\sqrt{2}}{2}i$

E.  $x = 2 \pm 2\sqrt{2}i$

F.  $x = \frac{-3 \pm \sqrt{14}i}{4}$

$$2x^2 + 4x + 3 = 0 \iff (x^2 + 2x + \frac{3}{2}) = 0$$

$$\iff (x + 1)^2 = -\frac{3}{2} + 1 = -\frac{1}{2}$$

$$\iff x + 1 = \pm \sqrt{-\frac{1}{2}} = \pm \frac{i}{\sqrt{2}} = \pm \frac{i\sqrt{2}}{2}$$

$$\iff x = -1 \pm \frac{\sqrt{2}}{2}i$$

6. At age 38, the age of Mr. Green is the sum of the ages of his three daughters. The age of the oldest daughter equals the sum of the ages of her two sisters. Also, the age difference between the oldest and youngest sisters is 14 years. What is the age of the youngest of the three daughters?

- A. 4 years  
 B. 2 years  
 C. 6 years  
 D. 5 years  
 E. Cannot be determined from the given data.  
 F. 3 years

let  $x =$  age of oldest daughter      Given  $x + y + z = 38$   
 $y =$  age of middle daughter      ;  $x - y - z = 0$   
 $z =$  age of youngest daughter       $x - z = 14$

augmented matrix  $\begin{pmatrix} 1 & 1 & 1 & 38 \\ 1 & -1 & -1 & 0 \\ 1 & 0 & -1 & 14 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 38 \\ 0 & -2 & -2 & -38 \\ 0 & -1 & -2 & -24 \end{pmatrix} \sim$   
 $\sim \begin{pmatrix} 1 & 1 & 1 & 38 \\ 0 & 1 & 1 & 19 \\ 0 & 1 & 2 & 24 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 38 \\ 0 & 1 & 1 & 19 \\ 0 & 0 & 1 & 5 \end{pmatrix}$

$\Rightarrow$   $z = 5$   
 $y = 14$   
 $x = 19$

7. For what value of  $k$  do the following two lines intersect?

$$\begin{aligned} x &= -3 + r & x &= 2 - 2t \\ y &= 3 + 2r & y &= 10 - 2t \\ z &= -r & z &= 1 + kt \end{aligned}$$

A.  $k=4$

B.  $k=0$

C.  $k=-2$

D.  $k=3$

E.  $k=6$

F.  $k=-1$

intersection  $\Leftrightarrow$

$$x = -3 + r = 2 - 2t \quad r + 2t = 5$$

$$y = 3 + 2r = 10 - 2t \quad \Leftrightarrow \quad 2r + 2t = 7$$

$$z = -r = 1 + kt \quad r + kt = -1$$

augmented  
matrix

$$\begin{pmatrix} 1 & 2 & 5 \\ 2 & 2 & 7 \\ 1 & k & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 5 \\ 0 & -2 & -3 \\ 0 & k-2 & -6 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 5 \\ 0 & 2 & 3 \\ 0 & k & -3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & -3 - 3k/2 \end{pmatrix}$$

is solvable  $\Leftrightarrow 0 = 3 + 3k/2$

$$\Leftrightarrow 0 = 1 + k/2$$

$$\Leftrightarrow k = -2$$

8. Let  $A = (a_{ij})$  be a  $4 \times 3$  matrix and  $B = (b_{ij})$  be a  $5 \times 3$  matrix. Give a formula for the  $(3,2)$ -element of  $AB^T$ .

A.  $(AB^T)_{32} = \sum_{k=1}^5 a_{2k} b_{3k}$

B.  $(AB^T)_{32} = \sum_{k=1}^3 a_{3k} b_{k2}$

C.  $(AB^T)_{32} = \sum_{k=1}^3 a_{2k} b_{3k}$

D.  $(AB^T)_{32} = \sum_{k=1}^3 a_{3k} b_{2k}$

E.  $(AB^T)_{32} = \sum_{k=1}^5 a_{3k} b_{2k}$

F.  $(AB^T)_{32} = \sum_{k=1}^3 a_{2k} b_{k3}$

$(3,2)$ -element of  $AC$  for  $A = (a_{ij})$ ,  $C = (c_{ij})$  is

$$\sum_{k=1}^3 a_{3k} c_{k2}$$

Now let  $C = B^T$ , i.e.  $c_{mn} = b_{nm}$ , so we get

$$\sum_{k=1}^3 a_{3k} b_{2k}$$

9. Use Cramer's Rule to solve for  $x_2$  in the following linear system:

$$\begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 3 \\ 5 & 3 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix}$$

A.  $x_2 = 14/9$

B.  $x_2 = -14/9$

C.  $x_2 = 10/3$

D.  $x_2 = -25/6$

E.  $x_2 = -10/3$

F.  $x_2 = 25/6$

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 3 \\ 5 & 3 & -6 \end{pmatrix}, \quad \det A = -3 \begin{vmatrix} 2 & 1 \\ 5 & 3 \end{vmatrix} - 6 \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix}$$

$$= -3(6-5) - 6(4-3) = -3-6 = -9$$

$$B_2 = \begin{pmatrix} 2 & 2 & 0 \\ 3 & 5 & 3 \\ 5 & -4 & -6 \end{pmatrix} = \text{obtained from } A \text{ by replacing the 2nd column of } A \text{ by the right side}$$

$$\det B_2 = -3 \begin{vmatrix} 2 & 2 \\ 5 & -4 \end{vmatrix} - 6 \begin{vmatrix} 2 & 2 \\ 3 & 5 \end{vmatrix} = -3(-8-10) - 6(10-6) =$$

$$= 3 \cdot 18 - 6 \cdot 4 = 54 - 24 = 30$$

$$x_2 = \frac{\det B_2}{\det A} = \frac{30}{-9} = -\frac{10}{3}$$

10. Let  $A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  and let  $B$  be any  $3 \times 4$  matrix. What can one say about row 2 of the product  $AB$ ?

- A. It is equal to row 1 of  $B$  multiplied by  $-2$ .  
 B. It is equal to row 2 of  $B$  minus two times row 1 of  $B$ .  
 C. It is equal to row 1 of  $B$  minus two times row 2 of  $B$ .  
 D. It is equal to row 3 of  $B$  minus two times row 2 of  $B$ .  
 E. It is equal to row 2 of  $B$  multiplied by  $-2$ .  
 F. It is equal to row 2 of  $B$ .

$$\begin{aligned}
 AB &= \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{pmatrix} \\
 &= \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ -2b_{11}+b_{21} & -2b_{12}+b_{22} & -2b_{13}+b_{23} & -2b_{14}+b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{pmatrix}
 \end{aligned}$$

$$\Rightarrow \text{row 2 of } AB = \text{row 2 of } B - 2 \cdot (\text{row 1 of } B)$$

11. [14 points]

(a) Indicate whether each of the following statements is True or False. Each correct answer is worth 2 points; a wrong answer is worth -1 point; no answer is worth 0 points.

F If the reduced row echelon (RREF) form of a matrix  $A$  has a row of 0's, then the linear system  $AX = 0$  will have infinitely many solutions.

T There exist square matrices  $A$  and  $B$  for which  $AB = BA$ .

F If  $A$  is an invertible matrix, its columns are linearly dependent.

F If  $A$  and  $B$  are invertible  $n \times n$  matrices, then  $(AB)^{-1} = A^{-1}B^{-1}$ .


Complete each of the following statements:

(b) If  $A = \begin{pmatrix} 3 & -2 \\ -5 & 4 \end{pmatrix}$  then  $A^{-1} = \begin{pmatrix} 2 & 1 \\ 5/2 & 3/2 \end{pmatrix}$

Note.  $\det A = 3 \cdot 4 - (-5)(-2) = 12 - 10 = 2$

So  $A^{-1} = \frac{1}{2} \begin{pmatrix} 4 & 2 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 5/2 & 3/2 \end{pmatrix}$

(c) Let  $P = (2, 1, 4)$  and  $Q = (6, 3, 1)$ . The point  $\frac{1}{3}$  the way from  $P$  to  $Q$  is  $\left(\frac{10}{3}, \frac{4}{3}, 3\right)$



$$\begin{aligned} \vec{OP} + \frac{1}{3} \vec{PQ} &= (2, 1, 4) + \frac{1}{3} [(6, 3, 1) - (2, 1, 4)] = \\ &= (2, 1, 4) + \frac{1}{3} (4, 2, -3) \\ &= \left(2 + \frac{4}{3}, 1 + \frac{2}{3}, 4 - 1\right) = \left(\frac{10}{3}, \frac{4}{3}, 3\right) \end{aligned}$$

(d) If  $A$  is a  $3 \times 3$  matrix and  $\det A = 4$ , then  $\det(2A^{-1}) = \underline{2}$

$$\det(2A^{-1}) = 2^3 (\det A)^{-1} = \frac{8}{4} = 2$$

12. [6 points] Answer each of the following in the blank provided:

(a) Suppose  $T: P_2 \rightarrow P_3$  is a linear map satisfying

$$T(1+x^2) = 1-x, \quad T(x+2x^2) = 1+x^2, \quad T(x^2) = 1+x^3.$$

Find  $T(x)$ .

$$x = (x+2x^2) - 2x^2$$

$$\Rightarrow T(x) = T(x+2x^2) - 2T(x^2)$$

$$= 1+x^2 - 2(1+x^3) = -1 + x^2 - 2x^3$$

Answer:  $T(x) = \underline{-1 + x^2 - 2x^3}$

(b) Let  $A$  be an  $n \times n$  matrix. Prove that  $A + A^T$  is symmetric.

$$(A + A^T)^T = A^T + (A^T)^T = A^T + A \Rightarrow A + A^T \text{ is symmetric}$$

(c) Give the definition of the *span* of a set of vectors  $\{v_1, v_2, v_3\}$  in a vector space  $V$ .

$$\text{span}\{v_1, v_2, v_3\} = \{av_1 + bv_2 + cv_3 \mid a, b, c \in \mathbb{R}\}$$

13. [10 points] Determine the values of  $b$  and  $c$  for which the system

$$\begin{cases} x + 2y + z = b \\ x + y + (1+b)z = b - c \\ x + y + z = 1 - 2c \end{cases}$$

has

- (a) exactly one solution
- (b) no solution
- (c) infinitely many solutions

In the case where there are infinitely many solutions, give the general solution.

$$\begin{aligned} \text{augmented matrix} & \begin{pmatrix} 1 & 2 & 1 & b \\ 1 & 1 & (1+b) & b-c \\ 1 & 1 & 1 & 1-2c \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & b \\ 0 & -1 & b & -c \\ 0 & -1 & 0 & 1-2c-b \end{pmatrix} \\ [1] & \sim \begin{pmatrix} 1 & 2 & 1 & b \\ 0 & 1 & -b & c \\ 0 & 0 & -b & 1-c-b \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & b \\ 0 & 1 & -b & c \\ 0 & 0 & b & b+c-1 \end{pmatrix} \end{aligned}$$

[2] (a) exactly one solution  $\Leftrightarrow b \neq 0$

[2] (b) no solution  $\Leftrightarrow b = 0, b+c-1 \neq 0$   
 $\Leftrightarrow b = 0, c \neq 1$

[3] (c)  $\infty$  many solutions:  $\Leftrightarrow b = 0, b+c-1 = 0$   
 $\Leftrightarrow b = 0, c = 1$

In this case,

$$\text{augmented matrix} \sim \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} z &= t \\ y &= 1 \\ x &= -2 - t \end{aligned}$$

14. [10 points] Consider the subspace  $U$  of  $M_{22}$  given by  $U = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}; a, b, c \in \mathbb{R} \right\}$  and the matrices  $v_1 = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$ ,  $v_2 = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$  and  $v_3 = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$ . Is  $\{v_1, v_2, v_3\}$  a basis for  $U$ ? Either prove that it is, or show that it isn't. Justify your answer.

The matrices  $v_1, v_2, v_3$  all lie in  $U$

Spanning set: We need to find  $x, y, z$  such that

$$x \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} + y \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} + z \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

for arbitrary  $a, b, c \in \mathbb{R}$ , i.e.

$$\begin{pmatrix} x+y+2z & 3x-y+2z \\ 0 & 2x+2y+z \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \iff \begin{cases} x+y+2z = a \\ 3x-y+2z = b \\ 2x+2y+z = c \end{cases}$$

$$\text{i.e.} \quad \begin{pmatrix} 1 & 1 & 2 & a \\ 3 & -1 & 2 & b \\ 2 & 2 & 1 & c \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 & a \\ 0 & -4 & -4 & b-3a \\ 0 & 0 & -3 & c-2a \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 & a \\ 0 & 1 & 1 & \frac{3a-b}{4} \\ 0 & 0 & 1 & \frac{2a-c}{3} \end{pmatrix}$$

system is uniquely solvable  $\Rightarrow v_1, v_2, v_3$  is not only a spanning set but the coefficients  $x, y, z$  are unique, i.e.  $\{v_1, v_2, v_3\}$  is a basis

Other methods I) Calculation above shows that  $U = \text{span}\{v_1, v_2, v_3\}$ .

Now check that  $\{v_1, v_2, v_3\}$  is linearly independent (a special case of calculation above:  $a = b = c = 0$ )

II)  $U = \text{span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ , and  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$  are linearly independent, so  $\dim U = 3$ . It is then sufficient to verify that  $v_1, v_2, v_3$  are linearly independent (or that  $U = \text{span}\{v_1, v_2, v_3\}$ )

15. [10 points] (a) Show that  $U = \{p(x) \in P_3; 2p(0) = p(2)\}$  is a subspace of  $P_3$ .  
 (b) Find a basis of  $U$ , and determine  $\dim U$ .

[4] (a) For  $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$  we have

$$2p(0) = p(2) \Leftrightarrow 2a_0 = a_0 + 2a_1 + 4a_2 + 8a_3$$

$$\Leftrightarrow a_0 = 2a_1 + 4a_2 + 8a_3$$

$$\text{So } U = \{a_0 + a_1x + a_2x^2 + a_3x^3 : a_0 = 2a_1 + 4a_2 + 8a_3\}$$

$$= \{(2a_1 + 4a_2 + 8a_3) + a_1x + a_2x^2 + a_3x^3 : a_1, a_2, a_3 \in \mathbb{R}\}$$

$$= \{a_1(x+2) + a_2(x^2+4) + a_3(x^3+8) : a_1, a_2, a_3 \in \mathbb{R}\}$$

$$= \text{span}\{x+2, x^2+4, x^3+8\}$$

As a span,  $U$  is a subspace.

Other method. check conditions of subspace test

[6] (b) We have seen in (a) that  $U = \text{span}\{x+2, x^2+4, x^3+8\}$

We check that  $x+2, x^2+4, x^3+8$  are linearly independent:

$$0 = a_1(x+2) + a_2(x^2+4) + a_3(x^3+8)$$

$$= (2a_1 + 4a_2 + 8a_3) + a_1x + a_2x^2 + a_3x^3$$

$$\Leftrightarrow 2a_1 + 4a_2 + 8a_3 = 0 = a_1 = a_2 = a_3, \text{ so } x+2, x^2+4, x^3+8$$

are linearly independent, and hence form a basis of  $U$

$$\dim U = 3$$

16. [10 points] (a) Find all eigenvalues of the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Is  $A$  diagonalizable? Please justify your answer.

(b) The eigenvalues of the matrix

$$B = \begin{pmatrix} -5 & 0 & 6 \\ -3 & 1 & 3 \\ -3 & 0 & 4 \end{pmatrix}$$

are  $\lambda_1 = 1$  and  $\lambda_2 = -2$ . Find all eigenspaces of  $B$ . Is  $B$  diagonalizable? Please justify your answer.

[4]

(a)

Characteristic  
polynomial

$$c_A(x) = \det(xI_3 - A) = \begin{vmatrix} x & 0 & -1 \\ 0 & x & 0 \\ -1 & 0 & x \end{vmatrix} =$$

$$= x \begin{vmatrix} x & -1 \\ -1 & x \end{vmatrix} = x(x^2 - (-1)(-1)) = x(x^2 - 1) = x(x-1)(x+1)$$

The eigenvalues of  $A$  are the roots of  $c_A(x)$ , hence  $A$  has the three distinct eigenvalues  $0, 1, -1$ .

Since the  $3 \times 3$  matrix  $A$  has 3 distinct eigenvalues, it is diagonalizable (§ 6.2 Thm 3)

[6] (b) (i)  $E_1(B) = \text{null}(I - B)$ :

$$I - B = \begin{pmatrix} 6 & 0 & -6 \\ 3 & 0 & -3 \\ 3 & 0 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ i.e. } x=1$$

$$\Rightarrow E_1(B) = \left\{ \begin{pmatrix} x \\ y \\ x \end{pmatrix} : x, y \in \mathbb{R} \right\} = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}, \text{ 2-dimensional}$$

(ii)  $E_{-2}(B) = \text{null}(-2I - B)$

$$-2I - B = \begin{pmatrix} 3 & 0 & -6 \\ 3 & -3 & -3 \\ 3 & 0 & -6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 \\ 1 & -1 & -1 \\ 1 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

[more work space on next page...]

$$\begin{aligned} & x = 2z \\ \Rightarrow & y = z \quad \rightarrow \quad E_{-2}(B) = \left\{ \begin{pmatrix} 2z \\ z \\ z \end{pmatrix} : z \in \mathbb{R} \right\} = \text{span} \left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\} \\ & = \mathbb{R} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \end{aligned}$$

B is diagonalizable since  $\mathbb{R}^3$  has a basis of eigenvectors of B, namely  $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

(Not required; for  $P = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$  we have  $P^{-1}BP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ )